Linear Vector Space

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1 Linear Vector Space Definition

Let V be a set on which two operations, addition and scalar multiplication, have been defined. If u and v are in V, the sum of u and v is denoted by $u + v$, and if c is a scalar, the scalar multiple of u by c is denoted by $c\mathbf{u}$. If the following axioms hold for all $\mathbf{u}$,$\mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and d, then V is called a *vector space* and its elements are called vectors.

10 Axioms of a Vector Space
1. $\mathbf{u} + \mathbf{v}$ is in V (closure under addition)
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (commutativity)
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ (associativity)
4. There exists an element $\mathbf{0}$ in V, called a **zero vector**, such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is an element $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u})=\mathbf{0}$.
6. $c\mathbf{u}$ is in V (closure under scalar multiplication)
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ (distributivity)
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ (distributivity)
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$
10. $1\mathbf{u} = \mathbf{u}$

2 Example

Determine whether the given set is a vector space:

Let V be the set of all positive real numbers x with $x + x' = xx'$, and $kx = x^k$

1. If x and y are positive reals, so is $x + y = xy$ (closure under addition)
2. $x + y = xy = yx = y + x$ (commutativity)
3. $x + (y + z) = x(yz) = (xy)z = (x + y) + z$ (associativity)
4. $1 + x = 1x = x = x1 = x + 1$ for all positive reals x (zero vector)
5. $x + (\frac{1}{x}) = x(\frac{1}{x}) = 1 = \mathbf{0} = 1 = (\frac{1}{x})x = (\frac{1}{x}) + x$ $c\mathbf{u}$ is in V (closure under scalar multiplication)
6. If k is real and x is a positive real, then $kx = x^k$ is again a positive real.
7. $k(x + y) = (xy)^k = x^k y^k = kx + ky$ (distributivity)
8. $(c + d)x = x^{c+d} = x^c x^d = cx + dx$ (distributivity)
9. $c(dx) = (dx)^c = (x^d)^c = x^{dc} = x^{cd} = (cd)x$
10. $1x = x^1 = x$

3 Linear Combination

A vector v is a *linear combination* of vectors $v_1, v_2, ..., v_k$ if there are scalar coefficients $c_1, c_2, ..., c_k$ such that $c_1v_1 + c_2v_2 + c_kv_k = v$

4 Example of a linear combination

u= $[1, 2, -1]$ v= $[6, 4, 2]$ in R^3.

Show that w= $[9, 2, 7]$ **is a linear combination of** u **and** v.

$c_1[1, 2, -1] + c_2[6, 4, 2] = [9, 2, 7]$

$[c_1, 2c_1, -1c_1] + [6c_2, 4c_2, 2c_2] = [9, 2, 7]$

$c_1 + 6c_2 = 9$

$2c_1 + 4c_2 = 2$

$-c_1 + 2c_2 = 7$

$c_1 = -3$

$c_2 = 2$

5 Linear Dependence

A set of vectors $v_1, v_2, ...v_k$ is *linearly dependent* if there are scalars $c_1, c_2, ..., c_k$ at least one of which is not zero, such that $c_1v_1 + c_2v_2 + ...c_kv_k = 0$

ie We say that a set of vectors is **linearly dependent** if one of them can be written as a linear combination of the others.

A set of vectors which is not linearly dependent is **linearly independent**.

Example:

Let **u**= $[1, 0, 3]$ **v**= $[-1, 1, -3]$ **w**= $[1, 2, 3]$

$3[1,0,3] + 2[-1,1,-3] - [1,2,3] = 0$
u, v, and **w** are linearly dependent, since
3**u** + 2**v** - **w** = 0 in fact, all the scalars are nonzero.

Definition: If $S = v_1, v_2, ..., v_k$ is a set of vectors in a vector space V, then the set of all linear combinations of $v_1, v_2, ..., v_k$ is called the *span* of $v_1, v_2, ..., v_k$.

The Basis Theorem: If a vector space V has a basis with n vectors, then every basis for V has exactly n vectors.

Definition: A vector space V is called *finite dimensional* if it has a basis consisting of finitely many vectors. The *dimension* of V, denoted by dim V, is the number of vectors in a basis for V. The dimension of the zero vector space **0** is defined to be zero. A vector space that has no finite basis is called *infinite-dimensional.*

Vector Addition: $u + v = [u_1 + v_1, u_2 + v_2, ..., u_n + v_n]$

Scalar Multiplication: $c\mathbf{v} = c[v_1, v_2] = [cv_1, cv_2]$