Determinants and Solutions of Linear Systems of Equations

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1 Introduction

In this paper, we will study determinants and solutions of linear systems of equations in some detail. We will learn the basics for each and expand on them.

2 Determinants

A determinant is a mathematical object which is very useful in the analysis and solution of systems of linear equations. Determinants are only defined for square matrices. A square matrix has horizontal and vertical dimensions that are the same (i.e., an *nxn* matrix). The difference between the form of a matrix and a determinant of a matrix is that a determinant is displayed using straight lines in-place of the square brackets. The determinant is a scalar quantity, which means a one-component quantity. The determinant is most often used to:

- test whether or not a matrix has an inverse
- test for linear dependence of vectors (in certain situations)
- test for existence/uniqueness of solutions of linear systems of equations

3 An *nxn* Matrix

In an *nxn* matrix, we follow certain rules for the appearance of the matrix. We let v_i symbolize the i^{th} row. $(a_{i1}, a_{i2}, ..., a_{in})$. If this row were to be multiplied by α, the the row would appear to be $(\alpha a_{i1}, \alpha a_{i2}, ..., \alpha a_{in})$. Also, if two rows were added, the i^{th} row and the j^{th} column, we would have $(a_{i1} + a_{j1}, a_{i2} + a_{j2}, ..., a_{in} + a_{jn})$. A unit matrix appears with the following rows: (1,0,0,...,0),(0,1,0,...,0),...,(0,0,0,...,1). Also, the letters $e_1, ..., e_n$ describe a unit row.

In an *nxn* matrix there are a few forms in which a determinant is recognized. First of all the determinant symbolizes the function of the n^2 variables $a_{ij}(i, j = 1, 2, ..., n)$. The determinant for this function can be written as:

$$D = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = |a_{ij}| = D(v_1, v_2, ...v_n)$$

4 Properties of Determinants

1. $D(v_1, v_2, ..., v_i, ..., v_n) = D(v_1, v_2, ..., v_i + v_j, ...v_n)$ $(i \neq j)$ (**Invariance**)
2. $D(v_1, v_2, ..., \alpha v_i, ..., v_n) = \alpha D(v_1, v_2, ..., v_i + v_j, ...v_n)$ (**Homogeneity**)
3. $D(e_1, e_2, ..., e_n) = 1$ (**Normalization**)

5 Rules for Determinants

Determinants are either written as $|A|$ or $detA$. Now say that we delete the i^{th} row and the j^{th} column a $(n-1)x(n-1)$ submatrix A_{ij} is formed. The determinant of this submatrix is the *minor* element of a_{ij}. The *cofactor* of a_{ij} is $(-1)^{i+j}|A_{ij}|$. We also write the cofactor as A^*_{ij}. The following are some rules for determinants.

a. $|A| = |A'|, A' = transposeof A = (a_j i))$

b. If two rows (or columns) of A are interchanged, producing a matrix A_1, then $|A| = -|A|$

c. If two rows (or columns) of A are identical, then —A—=0

d. If a row (or column), v, of A is replaced by kv producing a matrix A_1, then $|A|$=$k|A|$.

e. If a scalar multiple kv, of the ith row (or column) is added to the jth row (or column) v_j, $(i \neq j)$ and the matrix A_1 results, then $|A| = |A_1|$.

f. A determinant may be evaluated in terms of cofactors: $|A| =$

$$\sum_{i=1}^{n} a_{ij} A^*_{ij} \;\; 1 \leq j \leq n$$

$$= \sum_{i=1}^{n} a_{ij} A^*_{ij} \;\; 1 \leq i \leq n$$

6 $2X2$ Matrix

The most basic determinant is found using a $2x2$ matrix in the form

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The determinant of a $2x2$ matrix is found using the following formula:

$$|A| = det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

7 Example 1

$2x2$ **Matrix** Using the matrix

$$A = \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$$

the determinant would be

$$|A| = det(A) = \begin{vmatrix} 4 & 5 \\ 2 & 3 \end{vmatrix} = 4 * 3 - 5 * 2 = 2$$

8 $3x3$ Matrix

The next kind of matrix studied is the $3x3$ matrix. The form of this matrix is

$$B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

The determinant of a $3x3$ matrix is found using the following formula:

$$|B| = det(B) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

9 Example 2

$3x3$ **Matrix** Using the matrix

$$B = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 0 & 2 & 4 \end{bmatrix}$$

the determinant would be

$$|B| = det(B) = \begin{vmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 0 & 2 & 4 \end{vmatrix} = 1 \begin{vmatrix} 4 & 6 \\ 2 & 4 \end{vmatrix} - 3 \begin{vmatrix} 2 & 6 \\ 0 & 4 \end{vmatrix} + 5 \begin{vmatrix} 2 & 4 \\ 0 & 2 \end{vmatrix}$$

10 Solution of Linear Systems of Equations

Consider the system of n linear equations in n unknowns $x_1, x_2, ..., x_n$

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \ (i = 1, 2, ..., n)$$

11 Cramer's Rule

The above formula is Cramer's Rule. It states that if $|A| = |a_{ij}| \neq 0$, then

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \ (i = 1, 2, ..., n)$$

possesses a unique solution given by

$$x_r = \frac{\sum_{i=1}^{n} A_{ir}^{*} b_i}{|A|} \ (r = 1, 2, ..., n)$$

Cramer's rule is used to solve a set of n linear equations in n unknowns. It uses determinants to obtain the solution.

12 The Alternative Theorem

This formula that results from Cramer's Rule is the Alternative Theorem. Also described as the homogeneous system

$$\sum_{j=1}^{n} a_{ij}x_j = 0 \ (i = 1, 2, ..., n)$$

possesses a non-trivial solution (i.e., a solution other than $x_1 = x_2 = ... = x_n = 0$) if and only if $|A|$=0. If for a fixed $A = (a_{ij})$ there are solutions to the non-homogeneous system

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \ (i = 1, 2, ..., n)$$

for every selection of the quantities b_i, then $|A| \neq 0$ and the homogeneous system had only the trivial solution.