Differentiable Functions

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Abstract

In this paper I will briefly discuss one of the two fundamental operations of calculus-differentiation.

Contents

1 Introduction

In this paper I will give a definition of the derivative of a function, state some basic rules for differentiation, and give some examples of functions that are and are not differentiable

2 Definition

The **derivative** of f at x is give by

$$f'(x) = \lim_{\Delta x \mapsto 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

provided the limit exists.

Note: The process of finding the derivative of a function is called *differentiation*. A function is *differentiable* at x if its derivative exists at s and *differentiable* on an open interval (a, b) if it is differentiable at every point in the interval.

Other notations for the derivative of f include: $D_y[f(x)]$, $\frac{dy}{dx}$, or y'. The last two are used when the rule for f is written in the form $y = f(x)$.

3 Calculating the Derivative

The calculation of the derivative of f is facilitated using the following four-step process.

1. Compute $f(x + \Delta x)$.
2. Form the *difference* $f(x + \Delta x) - f(x)$.
3. Form the *quotient* $\frac{f(x+\Delta x)-f(x)}{\Delta x}$.
4. Compute $f'(x) = \lim_{\Delta x \mapsto 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$.

Example 3.1 *Let $f(x) = x^2$. Compute $f'(x)$.*

$$\begin{aligned}
f(x+\Delta x) &= (x+\Delta x)^2 = x^2 + 2x\Delta x + \Delta x^2 \\
f(x+\Delta x) - f(x) &= x^2 + 2x\Delta x + \Delta x^2 - x^2 = 2x\Delta x + \Delta x^2 = \Delta x(2x + \Delta x) \\
\frac{f(x+\Delta x) - f(x)}{\Delta x} &= \frac{\Delta x(2x+\Delta x)}{\Delta x} = 2x + \Delta x \\
f'(x) &= \lim_{\Delta x \mapsto 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \mapsto 0}(2x + \Delta x) = 2x
\end{aligned}$$

4 Differentiability and Continuity

In practical applications, you encounter functions that fail to be *differentible*-that is, do not have a derivative at certain values in the domain of the function f. It can be shown that a continuous function f fails to be differentiable at a point when the graph of f makes and abrupt change of direction at that point. The following alternative limit form of the derivative is useful un investigating the relationship between differentiability and continuity. The derivative of f at c is

$$f'(c) = \lim_{x \mapsto c} \frac{f(x) - f(c)}{x - c}$$

provided this limit exists. Note that the limit in this alternative form requires that the one-sided limits

$$\lim_{x \mapsto c^+} \frac{f(x) - f(c)}{x - c} \text{ and } \lim_{x \mapsto c^-} \frac{(f(x) - f(c)}{x - c}$$

exists and are equal. These one-sided limits are called the **derivatives from the left and from the right**, respectively.

Example 4.1 *The function*

$$f(x) = |x - 2|$$

is continuous at $x = 2$*. However, the one-sided limits*

$$lim \frac{f(x) - f(2)}{x - 2} = lim \frac{|x - 2| - 0}{x - 2} = -1 \quad and \quad lim \frac{f(x) - f(2)}{x - 2} = lim \frac{|x - 2| - 0}{x - 2} = 1$$

are not equal. Therefore, f *is not differentiable at* $x = 2$*.*

Theorem 4.1 *Differentiability Implies Coninuity*
If f *is differentiable at* $x = c$*, then* f *is continuous at* $x = c$*.*

Proof: You can prove that f is continuous at $x = c$ by showing that $f(x)$ approaches $f(c)$ as $x \to c$. To do this, use the differentiability of f at $x \to c$ and consider the following limit:

$$\begin{aligned} lim[f(x) - f(c)] &= lim[(x - c)(\frac{f(x) - f(c)}{x - c})] \\ &= [lim(x - c)][lim \frac{f(x) - f(c)}{x - c}] \\ &= (0)[f'(c)] \\ &= 0 \end{aligned}$$

Because the difference $f(x) - f(c)$ approaches zero as $x \to c$, you can conclude that $lim f(x) = f(c)$. Therefore, f is continuous at $x = c$.

q.e.d.

Note: If a function is differentiable at $x = c$, then it is continuous at $x = c$. Hence, differentiability implies continuity. It is possible for a function to be continuous at $x = c$ and *not* be differentiable at $x = c$. Hence, continuity does not imply differentiability.

5 A few Theorems for Differentiable Functions

There are many theorems related to differentiable functions. We already saw, for example, that differentiablity implies continuity. Other important theorems are *Rolle's Theorem* and its generalized form.

Theorem 5.1 *Rolle's Theorem Let* f *be continuous on the closed interval* $[a, b]$ *and differentiable on the open interval* (a, b)*. If* $f(a) = f(b)$ *then there is at least one number* c *in* (a, b) *such that* $f'(c) = 0$*.*

Example 5.1 *Suppose* $f(x) = x^4 - 2x^2$ *and we consider the interval* $[-2, 2]$*. Does Rolle's Theorem apply, and if so find the points guaranteed to exist by the theorem.*

Figure 1: Graph of $f(x) = x^4 - 2x^2$

To begin, note that f is continuous on the interval $[-2, 2]$ and differentiable on the interval $(-2, 2)$. Moreover, because $f(-2) = 8 = f(2)$ you can conclude that there exists at least one c in $(-2, 2)$ such that $f'(c) = 0$. Setting the derivative equal to 0 produces

$$f'(x) = 4x^3 - 4x = 0$$
$$4x(x^2 - 1) = 0$$
$$x = 0, 1, -1$$

Theorem 5.2 *Generalized Rolle's Theorem*

- f *once differentiable,* $f(a_1) = f(a_2) \Rightarrow f'(c_1) = 0$ *for some* c_1
- f *twice differentiable,* $f(a_1) = f(a_2) = f(a_3)$
  - $\Rightarrow f'(c_1) = 0$ *and* $f'(c_2) = 0$
  - $\Rightarrow f''(d_1) = 0$ *for some* d_1
- f *thrice differentiable,* $f(a_1) = f(a_2) = f(a_3) = f(a_4)$
  - $\Rightarrow f'(c_1) = 0$, $f'(c_2) = 0$, *and* $f'(c_3) = 0$
  - $\Rightarrow f''(d_1) = 0$ *and* $f''(d_2) = 0$
  - $\Rightarrow f'''(e_1) = 0$ *for some* e_1

Example 5.2 *let* $f(x) = x^4 - 2x^2$ *on* $[-1.3, 1.3]$ *then by the Generalized Rolle's Theorem,* $f(1.3) = -0.5239 = f(-1.3)$.

$$f'(x) = 4x^3 - 4x = 0$$
$$4x(x^2 - 1) = 0$$
$$x = 0, 1, -1$$

Thus there are three points where $f'(x) = 0$.

$$f''(x) = 12x^2 - 4 = 0$$
$$12x^2 = 4$$
$$x = \frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}$$

Thus there are two points where $f''(x) = 0$

$$f'''(x) = 24x = 0$$
$$x = 0$$

Thus there is one point where $f'''(x) = 0$

Now we saw that functions that are differentiable more than once could be useful, we define new classes of functions as follows.

Definition 5.1 *The Space of C^n Functions If $f(x)$ is n times differentiable on $[a,b]$ and if $f^n(x)$ is itself continuous on $[a,b]$, we shall write $f(x) \in C^n[a,b]$.*

Note:$C^n[a,b]$ is a linear space of functions.

Example 5.3 *Here is a function that is continuous but not differentiable (i.e. C^0):*

Let $f(x) = |x|$ and we consider the interval $[-1,1]$. The function is certainly continuous, but not differentiable at $x = 0$.

Example 5.4 *Here is a function that is differentiable but f' is not continuous (i.e. again C^0):*

$$f(x) = \begin{cases} x^2 sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Example 5.5 *The function $f(x) = x^{\frac{4}{3}}$ is a function that is in C^1 but not C^2.*

$$f(x) = x^{\frac{4}{3}}, x \in \Re$$

$$f'(x) = \frac{4}{3} x^{\frac{1}{3}} \Rightarrow f \in C'$$

$$f''(x) = \frac{4}{3} \cdot \frac{1}{3} x^{\frac{-2}{3}}$$

$$f''(x) = \frac{4}{9} \cdot \frac{1}{x^{\frac{2}{3}}}.$$

The function is not defined at $x = 0$, and thus not in C^2.

6 Taylor's Theorem

Now our basic question is this: if a function f is n-times differentiable, i.e. $f \in C^n$, then what does the function have to do with its derivatives, if anything? Or in other words, can I use the derivatives to approximate the function. The answer is yes, and Taylor's theorem helps us do so.

Theorem 6.1 *Taylor's Thereom Let $f(x) \in C^{n+1}[a,b]$ and let $x_0 \in [a,b]$, then*

$$f(x) \approx f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\ldots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\frac{1}{n!}\int_{t=x_0}^{t=x} f^{(n+1)}(t)(x-t)^n dt.$$

Theorem 6.2 *Taylor's Theorem with Exact Remainder Let* $f(x) \in C^n[a,b]$ *and let* $f^{(n+1)}(x)$ *exists in* (a,b)*. Then there exists a* ζ *with* $a < \zeta < b$ *such that*

$$f(b) = f(a)+f'(a)(b-a)+\frac{f''(a)}{2!}(b-a)^2+\ldots+\frac{f^{(n)}(a)}{n!}(b-a)^n+\frac{f^{(n+1)}\zeta}{(n+1)!}(b-a)^{n+1}.$$

Example 6.1 *Let* $f(x) = e^x \in C^5$ *for* $x = 2$ *, on* $[-1,1]$*.* $f(x)$ *is differentiable infinitely many times and since* $f^{(n)}(x) = e^x$ *for all* n*,* $f^{(n)}x_0 = 1$*. The Taylor expansion is given by*

$$e^x = 1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\frac{1}{120}x^5+\frac{e^{\zeta_5}}{720}x^6 = 7.26667+\textit{some arbitrary number}$$

From Taylor's Theorem, it is now easy to obtain a polynomial approximation by deleting the remainder term from the formula. Now we have

$$e^x = 1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\frac{1}{120}x^5 = 7.26667$$

$e^2 = 8.26667$*. Taylor's theorem gives a close approximation to the value of a function which we can not easily compute.*