Lipschitz Functions

Lorianne Ricco

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Definition 1 *Let $f(x)$ be defined on an interval I and suppose we can find two positive constants M and α such that*

$$|f(x_1) - f(x_2)| \leq M|x_1 - x_2|^\alpha \text{ for all } x_1, x_2 \in I.$$

Then f is said to satisfy a Lipschitz Condition of order α and we say that $f \in Lip(\alpha)$.

Example 1 *Take $f(x) = x$ on the interval $[a, b]$. Then*

$$|f(x_1) - f(x_2)| = |x_1 - x_2|$$

That implies that $f \in Lip(1)$.

Now take $f(x) = x^2$ on the interval $[a, b]$. Then

$$|f(x_1) - f(x_2)| = |{x_1}^2 - {x_2}^2| = |x_1 - x_2||x_1 + x_2| \leq M|x_1 - x_2|$$

with $M = 2max(|a|, |b|)$. Hence, again $f \in Lip(1)$.

The function $f(x) = 1/x$ on $(0, 1)$. Is it $Lip(1)$? How about $Lip(1/2)$? How about $Lip(\alpha)$?

Theorem 1 *$Lip(\alpha)$ is a linear space.*

Proof We will look at a part of this proof. Let $f, g \in Lip(\alpha)$.

$$(f + g)(x) \in Lip(\alpha)$$

Then,

$$|f(x) + g(x) - f(y) + g(y)| \leq M|x - y|^\alpha$$

If $f \in Lip(\alpha)$ it implies that

$$|f(x) - f(y)| \leq M_1|x - y|^\alpha$$

If $g \in Lip(\alpha)$ it implies that

$$|g(x) - g(y)| \leq M_2|x - y|^\alpha$$

If $(f + g) \in Lip(\alpha)$ it imples that

$$|(f + g)(x) - (f + g)(y)| \leq M_3|x - y|^\alpha$$

$$|f(x) + g(x) - f(y) - g(y)| =$$

$$|f(x) - f(y) + g(x) - g(y)| =$$

$$|f(x) - f(y)| + |g(x) - g(y)| =$$

By the triangle inequality,

$$|f(x) - f(y)| + |g(x) - g(y)| \leq$$

$$M_1|x - y|^\alpha + M_2|x - y|^\alpha = |M_1 + M_2||x - y|^\alpha$$

Theorem 2 *If $f \in Lip(\alpha)$ with $\alpha > 1$ then $f = constant$.*

Proof Left as homework for everyone.

1 Lipschitz and Continuity

Theorem 3 *If $f \in Lip(\alpha)$ on I, then f is continous; indeed, uniformly contiuous on I.*

Last time we did continuity with ϵ and δ. An alternative definition of continuity familar from calculus is: f is continuous at $x = c$ if:

- $f(c)$ exists
- $lim_{x \to c} f(x)$ exists
- $lim_{x \to c} f(x) = f(c)$

In order to be continuous, if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$.
Proof

$$|f(x) - f(c)| \leq M|x - c|^\alpha$$

$$lim_{x \to c}|f(x) - f(c)| \leq M lim_{x \to c}|x - c|^\alpha = 0$$

This implies

$$lim_{x \to c} f(x) = f(c)$$

How about continuous implies $Lip(\alpha)$?

2 Lipschitz and Differentiability

Theorem 4 *If $f \in Lip(\alpha)$, it may fail to be differentiable, but if it possesses a derivative satisfying $|f'(x)| \leq M$ then $f \in Lip(1)$.*

In order to be differentiable,

$$\lim_{x \to \infty} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

By the Mean Value Theorem,

$$\Rightarrow \frac{|f(x) - f(y)|}{|x - y|} = |f'(c)|, c \in (x, y)$$

This implies that

$$|f(x) - f(y)| = |f'(c)||x - y|$$

Now if

$$|f'(x)|$$

exsits and is bounded by M, then

$$|f(x) - f(y)| \leq M|x - y|$$

which implies $f \in Lip(1)$.