

# Linear Functionals and the Algebraic Conjugate Space

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In many problems, we must associate a number with a function extracted from a given class of functions. For example:

- to each function  $f(x)$  that has a continuous derivative on  $[a, b]$ , we may want to associate the number  $\int_a^b (1 + [f'(x)]^2)^{\frac{1}{2}} dx$ .
- to each function  $f(x, y)$  that is twice continuously differentiable over a closed bounded region  $B$ , we may have to form the number  $\int \int_B \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 dx dy$
- for the same function as above we associate, more simply,  $f(x_0, y_0)$  where  $(x_0, y_0) \in B$ .

Such an association is known as a *functional*.

**Definition 0.1** Let  $X$  be a linear vector space and to each  $x$  let there be associated a unique real (or complex) number designated by  $L(x)$ . If for  $x, y \in X$  and for all real (or complex)  $\alpha, \beta$  we have

$$L(\alpha x + \beta y) = \alpha L(x) + \beta L(y),$$

then  $L$  is called a linear functional over  $X$ .

**Example 0.1**  $X = C[a, b]$ , i.e. the elements of  $X$  are continuous functions  $f(x)$ . We define  $L$  as

$$L(f) = \int_a^b f(x) dx$$

To show this is a linear functional we must verify it is a functional and it is linear.

To see that it is a functional is easy: if  $f \in C[a, b]$  then  $f$  is continuous and therefore integrable over the interval  $[a, b]$ . Therefore the integral  $L(f) = \int_a^b f(x) dx$  exists, so that  $L$  associates to every element  $f$  a number.

To show that it is linear we need to verify that for two functions  $f$  and  $g$  in  $C[a,b]$  we have  $L(\alpha f + \beta g) = \alpha L(f) + \beta L(g)$ :

$$\begin{aligned} L(\alpha f + \beta g) &= \int_a^b \alpha f(x) + \beta g(x) dx = \\ &= \int_a^b \alpha f(x) dx + \int_a^b \beta g(x) dx = \\ &= \alpha \int_a^b f(x) dx + \beta \int_a^b g(x) dx = \\ &= \alpha L(f) + \beta L(g) \end{aligned}$$

which finishes the proof.

**Example 0.2** Let  $X = C[a, b]$  be as before but this time define  $L$  as:

$$L(f) = \int_a^b x^2 f(x) dx$$

It is easy to see - similar to above - that  $L$  is again a functional as well as linear.

**Example 0.3**  $X = C^2[a, b]$ .

$$L(f) = f''(a) + f'(b) - f\left(\frac{a+b}{2}\right)$$

Interpolation theory is concerned with reconstructing functions on the basis of certain functional information assumed known. In many cases, the functionals are linear.

Functionals can be added to one another and scalar products can be formed. If, for instance,  $f \in C^1[a, b]$  and

$$L_1(f) = \int_a^b f(x) dx \text{ and } L_2(f) = f'\left(\frac{a+b}{2}\right)$$

we can identify the functional

$$L(f) = \alpha \int_a^b f(x) dx + \beta f'\left(\frac{a+b}{2}\right)$$

with the expression  $\alpha L_1 + \beta L_2$ .  $L$  is itself a linear functional.

**Definition 0.2** Let  $X$  be a given linear space and let  $L_1$  and  $L_2$  be two linear functionals defined on  $X$ . The sum of  $L_1$  and  $L_2$  and the scalar product of  $\alpha$  and  $L_1$  are defined by

$$(a)(L_1 + L_2)(x) = L_1(x) + L_2(x), x \in X$$

$$(b)(\alpha L_1)(x) = \alpha L_1(x)$$

It is a simple matter to show that the set of all linear functionals defined on  $X$  combined by the above rules constitute a second linear space.

**Definition 0.3** Let  $X$  be a given linear space. The set of linear functionals defined on  $X$  and combined by Definition 0.2 forms a linear space called the **algebraic conjugate space** of  $X$  and denoted by  $X^*$ .  $X^*$ , then, has elements that are linear functionals. We can speak of linear combinations, linear independence, dimension, bases, etc., for linear functionals.

**Theorem 0.1** If  $X$  has dimension  $n$  then  $X^*$  has dimension  $n$  also.