

Analytic and Entire Functions

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1 What is a Complex Function?

We shall begin with a review of the basics.

Definition 1.1 (Function) *A function f is a rule that assigns to each element in a set A one and only one element in a set B .*

If f assigns the value b to the element a in A , we write

$$b = f(a)$$

and call b the image of a under f . As a result of this definition of function we sometimes refer to f as a mapping of A onto B .

We are not concerned with just any functions. We are concerned with complex valued functions of a complex variable, that is $z = x + iy$, where x is the real part and y is the imaginary part (both real-valued). For complex valued functions, if w denotes the value of the function f at the point z , we then write

$$w = f(z).$$

Also, just as the complex number z decomposes into real and imaginary parts, the complex function w also decomposes into real and imaginary parts, again both real-valued. This is customarily written

$$w = u(x, y) + iv(x, y),$$

with u and v denoting the real and imaginary parts respectively. Thus, a complex-valued function of a complex variable is, in essence, a pair of real functions of two real variables.

Example 1.1 Write the function $w = z^2 + 2z$ in the form $w = u(x, y) + iv(x, y)$.

Solution: By setting $z = x + iy$ we obtain

$$w = (x + iy)^2 + 2(x + iy) = x^2 - y^2 + i2xy + 2x + i2y.$$

Which then can be rewritten as

$$w = (x^2 - y^2 + 2x) + i(2xy + 2y).$$

■

2 Limits and Continuity of Complex Functions

The concepts of limits and continuity for complex functions are similar to those for real functions. Let's first examine the concept of the limit of a complex-valued function.

Definition 2.1 (Limit) Let f be a function defined in some neighborhood of z_0 , with the possible exception of the point z_0 itself. We say that the limit of $f(z)$ as z approaches z_0 is the number w_0 and write

$$\lim_{z \rightarrow z_0} f(z) = w_0,$$

or equivalently,

$$f(z) \rightarrow w_0 \text{ as } z \rightarrow z_0,$$

if for any $\epsilon > 0$ there exists a positive number δ such that

$$|f(z) - w_0| < \epsilon \text{ whenever } 0 < |z - z_0| < \delta.$$

This is essentially the same definition the we learned in calculus I when we were first learning limits.

The condition of continuity is as follows.

Definition 2.2 (Continuous) Let f be a function defined in a neighborhood of z_0 . Then f is continuous at z_0 if

$$\lim_{z \rightarrow z_0} f(z) = f(z_0).$$

In other words, for f to be continuous at z_0 , it must have a limiting value at z_0 , and this limiting value must be $f(z_0)$.

Furthermore, a function f is said to be continuous on a set S if it is continuous at each point of S .

The definitions of this section are almost identical to concepts introduced in Calculus I. In fact, the properties of limits and continuous functions that we learned to be true for real functions also hold true for complex-valued functions. Two such theorems are stated below.

Theorem 2.1 *If $\lim_{z \rightarrow z_0} f(z) = A$ and $\lim_{z \rightarrow z_0} g(z) = B$, then*

$$(i) \lim_{z \rightarrow z_0} (f(z) \pm g(z)) = A \pm B,$$

$$(ii) \lim_{z \rightarrow z_0} f(z)g(z) = AB,$$

$$(iii) \lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = \frac{A}{B}, \text{ if } B \neq 0.$$

Theorem 2.2 *If $f(z)$ and $g(z)$ are continuous at z_0 , then so are $f(z) \pm g(z)$ and $f(z)g(z)$. The quotient $\frac{f(z)}{g(z)}$ is also continuous at z_0 provided $g(z_0) \neq 0$.*

Here are some simple examples using these concepts of limits and continuity.

Example 2.1 *Find the limit as $z \rightarrow 2i$ of the function $f(z) = z^2 - 2z + 1$.*

Solution: Since $f(z)$ is continuous at $z = 2i$, we simply evaluate it there,

$$\lim_{z \rightarrow 2i} f(z) = f(2i) = 2(2i)^2 - 2(2i) + 1 = -3 - 4i.$$

■

Example 2.2 *Find the limit as $z \rightarrow 2i$ of the function $f(z) = \frac{z^2+4}{z(z-2i)}$.*

Solution: The function $f(z)$ is not continuous at $z = 2i$ because it is not defined there. However, for $z \neq 2i$ and $z \neq 0$ we have

$$\lim_{z \rightarrow 2i} f(z) = \frac{(z+2i)(z-2i)}{z(z-2i)} = \frac{z+2i}{z} = \frac{2i+2i}{2i} = \frac{4i}{2i} = 2.$$

■

3 Complex Differentiation

In general, a complex function of a complex variable, $f(z)$, is an arbitrary mapping from the xy -plane to the w -plane. A complex function is split into real and imaginary parts, u and v , and any pair $u(x, y)$ and $v(x, y)$ of two-variable functions gives us a complex function $u + iv$. However, notice there is something special about the pair

$$u_1(x, y) = x^2 - y^2, \text{ and } v_1(x, y) = 2xy,$$

as opposed to

$$u_2(x, y) = x^2 - y^2, \text{ and } v_2(x, y) = 3xy.$$

The difference is that the complex function $u_1 + iv_1$ treats $z = x + iy$ as a single "unit", because $x^2 - y^2 + i2xy = (x + iy)^2$. These are the types of functions that are complex differentiable.

Definition 3.1 *Let f be a complex-valued function defined in a neighborhood of z_0 . Then the derivative of f at z_0 is given by*

$$\frac{df}{dz}(z_0) \equiv f'(z_0) := \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z},$$

provided this limit exists. (Such an f is said to be differentiable at z_0).

The catch in this definition is that Δz is a complex number, so it can approach zero in many different ways. Even though this catch may make things seem slightly more difficult, the rules for differentiating real functions apply in the same way for complex-valued functions (as long as the complex-valued function is in a form where $z = x + iy$ is treated as a single unit).

Example 3.1 *Show that, for any positive integer n ,*

$$\frac{d}{dz} z^n = nz^{n-1}.$$

Solution: Using Definition 3.1 we have

$$\frac{(z + \Delta z)^n - z^n}{\Delta z} = \frac{nz^{n-1}\Delta z + \frac{n(n-1)}{2}z^{n-2}(\Delta z)^2 + \cdots + (\Delta z)^n}{\Delta z}.$$

Thus

$$\frac{d}{dz} z^n = \lim_{\Delta z \rightarrow 0} \left[nz^{n-1} + \frac{n(n-1)}{2}z^{n-2}\Delta z + \cdots + (\Delta z)^{n-1} \right] = nz^{n-1}.$$

■

Theorem 3.1 *If f and g are differentiable at z , then*

$$(f \pm g)'(z) = f'(z) \pm g'(z), \quad (1)$$

$$(cf)'(z) = cf'(z) \text{ for any constant } c, \quad (2)$$

$$(fg)'(z) = f(z)g'(z) + f'(z)g(z), \quad (3)$$

$$\left(\frac{f}{g}\right)'(z) = \frac{g(z)f'(z) - f(z)g'(z)}{g(z)^2} \text{ if } g(z) \neq 0. \quad (4)$$

$$(5)$$

If g is differentiable at z and f is differentiable at $g(z)$, then the chain rule holds:

$$\frac{d}{dz}f(g(z)) = f'(g(z))g'(z).$$

4 Analyticity

Now that we have a secure background we are ready to look at the theory of *analytic* functions.

Definition 4.1 *A complex-valued function $f(z)$ is said to be analytic on an open set G if it has a derivative at every point of G .*

Analyticity is a property defined over open sets, while differentiability could hold at one point only. If the phrase " $f(z)$ is analytic at the point z_0 " is used it means that $f(z)$ is analytic in some neighborhood of z_0 .

To show that a function is analytic we use the following equations which must hold at $z_0 = x_0 + iy_0$:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

known as the *Cauchy-Riemann* equations.

Theorem 4.1 *A necessary condition for a function $f(z) = u(x, y) + v(x, y)$ to be differentiable at a point z_0 is that the Cauchy-Riemann equations hold at z_0 . Consequently, if f is analytic in an open set G , then the Cauchy-Riemann equations must hold at every point of G .*

Example 4.1 *Show that the function $f(z) = (x^2 + y) + i(y^2 - x)$ is not analytic at any point.*

Solution: Since $u(x, y) = x^2 + y$ and $v(x, y) = y^2 - x$ we have

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial v}{\partial y} = 2y,$$

$$\frac{\partial u}{\partial y} = 1, \quad \frac{\partial v}{\partial x} = -1.$$

Hence the Cauchy-Riemann equations are not satisfied and $f(z)$ is nowhere analytic. ■

Example 4.2 If $f(z) = x^3 - 3xy^2 + i(3x^2y - y^3)$ then determine where, if at all, the function is analytic. If it is analytic, find the complex derivative of f .

Solution: Since $u(x, y) = x^3 - 3xy^2$ and $v(x, y) = 3x^2y - y^3$ we have

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2, \quad \frac{\partial v}{\partial y} = 3x^2 - 3y^2,$$

$$\frac{\partial u}{\partial y} = -6xy, \quad \frac{\partial v}{\partial x} = 6xy.$$

Hence the Cauchy-Riemann equations are satisfied and $f(z)$ is analytic. The complex derivative of $f(z)$ is $f'(z) = 3x^2 - 3y^2 + i(6xy) = 3z^2$. ■

Theorem 4.2 (Cauchy's Integral Theorem) If f is analytic on a simply connected domain D and Γ is any loop (closed contour) in D then $\int_{\Gamma} f(z)dz = 0$.

Given f analytic inside and on the simple closed contour Γ , we know from Theorem 4.2 that $\int_{\Gamma} f(z)dz = 0$. However, if we consider the integral $\int_{\Gamma} \frac{f(z)}{z-z_0} dz$, where z_0 is a point in the interior of Γ , then the integral is not going to be zero because the integrand has a singularity inside the contour Γ . As a result, we have Cauchy's Integral Formula.

Theorem 4.3 Let Γ be a simple, closed, positively oriented contour. If f is analytic in some simply connected domain D containing Γ and z_0 is any point inside Γ then

$$f(z_0) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0).$$

Example 4.3 Find $\oint_C \frac{\cos(z)}{z+2+i} dz$, where C is the square with corners at 0, 1, $1+i$, and i .

Solution: First we must rewrite the problem in the form $\frac{f(z)}{z-z_0}$.

$$\oint_C \frac{\cos(z)}{z + 2 + i} dz$$

$$\oint_C \frac{\cos(z)}{z - (-2 - i)} dz$$

From this it is easy to see that $z_0 = -2 - i$ which is not included in C so it does not pose a problem and by Theorem 4.2

$$\oint_C \frac{\cos(z)}{z + 2 + i} dz = 0.$$

■

Example 4.4 Find $\oint_C \frac{e^z}{(z+5)^3(z-i)} dz$, where C is the circle centered at the origin of radius 2.

Solution: Again in this example we must rewrite the problem in the form $\frac{f(z)}{z-z_0}$.

$$\begin{aligned} &= \oint_C \frac{e^z}{(z+5)^3(z-i)} dz \\ &= \oint_C \frac{\frac{e^z}{(z+5)^3}}{(z-i)} dz \end{aligned}$$

In this case it is easy to see that $f(z) = \frac{e^z}{(z+5)^3}$ and $z_0 = i$. In this case $z_0 = i$ is included in C so Theorem 4.3 can be applied.

$$\begin{aligned} &= \oint_C \frac{\frac{e^z}{(z+5)^3}}{(z-i)} dz \\ &= 2\pi i f(i) \\ &= 2\pi i \frac{e^i}{(i+5)^3} \end{aligned}$$

■

5 Entire

If $f(z)$ is analytic on the whole complex plane, then it is said to be *entire*.

Definition 5.1 A function $f(z)$ is called *entire* if it has a representation of the form

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \quad \text{valid for } |z| < \infty.$$

This class of functions is designated by E . E is a linear space.

Example 5.1 Some examples of entire functions are

$$\frac{\sin(z)}{z}, 2^z, \int_0^z e^{t^2} dt, \frac{1}{\Gamma(z)}.$$

Theorem 5.1 The function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is entire if and only if

$$\lim_{n \rightarrow \infty} |a_n|^{\frac{1}{n}} = 0.$$