Infinitely Differentiable Functions

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1 Introduction

I will discuss the section of infinitely differentiable functions, their properties, and some useful examples to show the difference between functions that are infinitely differentiable, and those that aren't.

2 Differentiable functions

Definition: : A real function is said to be $differentiable$ at a point if its derivative exists at that point. The notion of differentiablity can also be extended to complex functions (leading to the Cauchy-Riemann equations and the theory of holomorphic functions)

3 Infinitely Differentiable Functions

Definition: If $f(x) \in C^n$ on the interval $[a, b]$ for n = 0,1,2..., then f is called Infinitely Differentiable on [a,b]. We shall write $C^\infty[a, b]$ for the class of such functions.

Use the definition to find the Taylor series (centered at c) for the function

Example 3.1 $f(x) = e^x, where\ c = 0$

Example 3.2 $f(x) = e^{-2x}$

Example 3.3 $f(x) = cos(x), where\ c = \frac{\pi}{4}$

Example 3.4 $f(x) = lnx, where\ c = 1$

Example 3.5 $f(x) = \frac{1}{1+x^2}\ is\ C^\infty$

4 Taylor Series

Definition: : If a function f has derivatives of all orders at x=c, then the series

$$\sum_{n=0}^{\infty}(x) = f(c) + f'(c)(x-c) + ... + \frac{f^n(c)}{n!}(x-c)^n + ...$$

is called the Taylor series. If $c = 0$ then the series is a **Mclaurin Series for** f

$$\lim_{n\to\infty} \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1} = 0$$

for every x in I.

5 Summary of Taylor Series

$$fx) = \sum_{n=0}^{\infty} \frac{f^n(c)}{n!}(x-c)^n$$

the coefficients x_n are given by: $x_n = \frac{f^n(c)}{n!}$ where $f^n(c)$ is the nth derivative of

f evaluated at a point c.

The functions of class $C^\infty[a,b]$ form a linear spce. If $f \in C^\infty[a,b]$ and $x_0 \in [a,b]$ we may form the Taylor expansion. The Taylor expansion for a given x, this series may or may not converge. If it converges, it may or may not converge to $f(x)$.

Proof:

Facts to prove:

Example 5.1 f *is continuous*

Example 5.2 f *is differentiable and f'(0)=0*

$f^n(0) = 0$

Let $p(x), q(x) \in \mathbf{R}$ be polynomials and define

if $f \in C^\infty$, then we can certainly write a taylor series for f. However, it requires that this Taylor series actually converge (at least across some radius of convergence, such that the series converges absolutely for all real or complex numbers.)

$$f(x) = \begin{cases} e^{\frac{-1}{x^2}} & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Then $f \in C^\infty$, and for any n greater than or equal to 0, $f^{(n)}(0) = 0$. So the Taylor series around 0 is 0; since $f(x) > 0$ for all x not equal to 0, clearly it does not converge to f.

Example 5.3 *f is $C^\infty(\boldsymbol{R})$ and $f^n(0) = 0$*

q.e.d.