Hierarchy of Functions

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February 4, 2004

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1 Introduction

Functions satisfy different properties which allow us to learn more about them. More specifically, we will be focusing on varying types of functions that lie in a single real or complex plane. These functions are know as L^p functions, bounded functions, continuous functions, and furthermore, uniformly continuous functions. We will be defining what we mean by these terms as well as showing some examples to understand these properties.

2 L^p Functions

Definition: Let $p > 0$ and $|f(x)|^p$ be integrable over $[a, b]$. Then this function is known as $L^p[a, b]$. In simplier terms, an L^p function can be defined as a function where

$$\int_a^b |f(x)|^p \, dx$$

exists.

Example 2.1 *Let* $f(x) = \frac{1}{x}$ *on the interval* $[0.1]$*. Show if* ***(a)*** $f \in L^1$*,* ***(b)*** $f \in L^2$*, and* ***(c)*** $f \in L^{\frac{1}{2}}$ *on the interval* $[0,1]$*:*

(a) $\int_0^1 \left|\frac{1}{x}\right| dx = ln(x)$ *over* $[0,1]$*. We know that the* $ln(x)$ *from* $[0,1]$ *is undefined because the* $ln(0) = \infty$*.*

(b) $\int_0^1 \left|\frac{1}{x^2}\right| dx = -\frac{1}{x}$ *over* $[0,1]$*. We know that* $-\frac{1}{x}$ *from* $[0,1]$ *is undefined because* $-\frac{1}{0}$ *is undefined.*

(c) $\int_0^1 \left|\frac{1}{x^{\frac{1}{2}}}\right| dx = -\frac{1}{2}x^{\frac{1}{2}}$ *over* $[0,1]$*. We know that this integral evaluated at* $[0,1]$ *is equal to* $-\frac{1}{2}$*.*

Therefore, $f \notin L^1$ *and* $f \notin L^2$*, but* $f \in L^{\frac{1}{2}}$

3 Bounded Functions

Definition: Suppose S is a nonempty set. Then the function $S \rightarrow M$ is a **bounded function** if there exists a M such that $|f(x)| < S$ for all $x \in S$.

The set of all functions which are bounded on S is denoted by $B(S)$ and $B(S)$ is a linear space.

Example 3.1 *The set* $y = sin(x^2)$ *is bounded on* $-\infty < x < \infty$*.*

Example 3.2 *The Gamma Function* $y = \Gamma(x)$ *is not bounded on the interval* $0 < x \leq 1$ *and the interval* $1 \leq x < \infty$*.*

4 Continuous Functions

Definition: A function is **continuous** at a point c in its domain D if given any $\epsilon > 0$ there exists a $\delta > 0$ such that if $x \in D$ and $|x - c| < \delta$, then $|f(x) - f(C)| < \epsilon$.

Moreover, a function is continuous if it is continuous at every point in its domain D.

Example 4.1 *The function* $f(x) = 5x - 6$ *is continuous in its domain However, the function*

$$g(x) = \begin{cases} 1, & if\ x = rational \\ 0, & if\ x = irrational \end{cases}$$

is not a continuous function.

PICTURE GOES HERE

Figure 1: Not Uniformly Continuous

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Figure 2: Uniformly Continuous

5 Uniformly Continuous Functions

Definition: A function f with a domain D is **uniformly continuous** on D if for any $\epsilon > 0$ there exists a $\delta > 0$ such that if $s, t \in D$ and $|s - t| < \delta$ then $|f(s) - f(t)| < \epsilon$.

To understand greater what we mean by uniformly continuous, take a look at figures 1 and 2:

As you can see in figure 1, there is not one single δ on the x-axis that will work for any fixed ϵ on the y-axis. Therefore, the graph cannot be uniformly continuous.

As you can see in figure 2, for any ϵ on the y-axis there exists a single δ that will work. Therefore, the graph is uniformly continuous.

6 Compact Sets

In a specific case, both continuity and uniform continuity overlap.

Theorem 6.1 *A function which is continuous on a compact point set is uniformly continuous.*

To understand this theorem we first need to define what a compact set is.

Definition: A set S of real numbers is a **compact set** if every sequence in S has a subsequence that converges to an element contained in S.

This definition is not so clear, therefore the following proposition helps you to understand compact sets better.

Proposition 6.1 *A set S of real numbers is compact if and only if it is* ***closed*** *and* ***bounded****.*

Example 6.1 *In the earlier example above, we have already found that $f(x) = \frac{1}{x}$ on the interval $(0, \frac{1}{2})$ is a continuous function. However, this function is not uniformly continuous because it is not continuous on a compact point set.*

7 First Mean Value Theorem for Integrals

Definition: Let $f(x)$ be continuous on $[a, b]$, and let

$$F(x) = \int_a^x f(t)dt$$

Then, we know from the Fundamental Theorem of Calculus that $F'(x) = f(x)$. We also know from the Mean Value Theorem that there exists a $c \in (a, b)$ such that

$$\frac{F(b) - F(a)}{b - a} = F'(c) \Rightarrow F(b) - F(a) = F'(c)(b - a)$$

This implies that

$$\int_a^b f(t)dt = f(c)(b - a)$$

and is known as the **First Mean Value Theorem for Integrals**. And the number $f(c)$ is referred to as the weighted average value of f on the interval $[a, b]$.

Example 7.1 *Find the average value of $f(x) = 5 - 2x$ on the interval $[-1, 2]$ and find a point in the interval at which the function takes on this average value.*

Solution $\frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{2-(-1)}\int {}_1{}^2(5 - 2x)dx = \frac{1}{3}[5x - x^2]_{-}1^2 = \frac{1}{3}(6 - (-6)) = 4$

$$f(x) = 4 \Rightarrow 5 - 2x = 4 \Rightarrow -2x = -1 \Rightarrow x = \frac{1}{2}$$

Therefore the average value of $f(x)$ on $[-1, 2]$ is 4 and a point in the interval at which the function takes on this average value is $\frac{1}{2}$.

References

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