MATH 3912 - Assignment 5

1. Suppose $p \in \boldsymbol{P}_2$ and $p(x_0) = p'(x_0) = p''(x_0) = 0$. Show that p must be identically zero.

2. Suppose $p \in \boldsymbol{P}_2$ and suppose that $p(x_1) = p'(x_1) = 0$ and $p(x_2) = 0$ for some fixed numbers $x_1 \neq x_2$. Show that p must then be identically zero.

3. Show that if $p \in \boldsymbol{P}_n$ and p has a root of total multiplicity $n+1$ then p must be identically zero. Recall that p has a root of multiplicity $n+1$ at $x = a$ if $p(a) = p'(a) = \ldots = p^{(n)}(a) = 0$

4. If $p(x)$ is a polynomial of even degree, what is $lim_{x\to\infty} p(x)$ and $lim_{x\to-\infty} p(x)$? Evaluate the same limits in case p is a polynomial of odd degree.

5. If $f(x)$ is a polynomial then $lim_{n\to\infty} f^{(n)}(x) = 0$ for all x.

6. Show that $f(x) = 2^x$ can coincide with a polynomial at only a finite number of points.

7. Suppose f is real analytic for all x and $f^{(k)}(x) > 0$ for $k = 0, 1, \ldots$. Then f can not coincide with a polynomial infinitly often.

8. Suppose $X = C^1([a,b])$ and x_0 is a fixed point in $[a,b]$. Define $L(f) = f'(x_0)$ for all $f \in X$. Is L a linear functional?

9. Suppose $X = C^0([a,b])$. Define two functionals L and G via

$$L(f) = \int_a^b x^2 f(x)dx \text{ and } G(f) = \int_a^b (f(x))^2\, dx$$

   Which of these functions is linear?

10. Suppose $X = \boldsymbol{P}^n$, x_0 is a fixed point in $[a,b]$, and f is a continous function defined on the interval $[a,b]$. Define $L(p) = (f \circ p)(x_0)$. Is L a linear functional?

11. Suppose $X = C^1([a,b])$. Define $L(f) = f'$ for all $f \in X$. What is the range of this map L? Is the map L linear? Is L a linear functional?

12. Let X be the space of $n \times n$ matrices and define $L(A) = \det(A)$. Is L a linear functional?

13. Let X be the space of $n \times n$ matrices. For a matrix $A \in X$ define the trace of A as

$$trace(A) = trace \begin{pmatrix} a_{1,1} & a_{1,2} & \ldots & a_{1,n} \\ a_{2,1} & a_{2,2} & \ldots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \ldots & a_{n,n} \end{pmatrix} = a_{1,1} + a_{2,2} + \ldots + a_{n,n}$$

   Is $L(A) = trace(A)$ a linear operator?