MATH 3912 - Assignment 3

1. Recall that a function is $Lip(\alpha)$ if $f(x) - f(y)| \leq M|x-y|^{\alpha}$ for all x, y in the function's domain.
   (a) Is it true that a function f that is $Lip(1)$ is continuous? Prove it or give a counter-example.
   (b) Is it true that a function $f : D \mapsto R$ is continuous implies that f is $Lip(1)$? If not, give a counter-example.
   (c) Is it true that a function f that is $Lip(1)$ is uniformly continuous? Prove it or give a counter-example.
   (d) Is it true that if f is uniformly continuous it must be $Lip(1)$? If not, give a counter-example (you could try $f(x) = x^{\frac{1}{3}}$ in a neighborhood of 0)

2. Is the function $f(x) = x^2 \in Lip(1)$? How about the function $f(x) = \frac{1}{x}$ on the interval $(0, 1)$?

3. Show that if $f : [a, b] \mapsto R$ is a function that is differentiable and whose derivative is bounded on $[a, b]$ is $Lip(1)$.

4. Find a function $f : [0, 1] \mapsto R$ that is differentiable but not $Lip(1)$.

5. Find a function $f : [0, 1] \mapsto R$ that is $Lip(1)$ but not differentiable.

6. Show that if $f \in Lip(2)$ and f is differentiable then f must be constant. As a hint, consider the definition of derivative, then use the Lipschitz condition to conclude something about f'.

7. Prove that if $f \in Lip(\alpha)$ with $\alpha > 1$ and f differentiable then f must be constant. For extra credit, prove this statement when the differentiability condition is dropped.