Chapter 6

Linear Programming: The Simplex Method

Section 2

The Simplex Method: Maximization with Problem Constraints of the Form ≤

Example 1

We will solve the same problem that was presented earlier, but this time we will use the Simplex Method. We wish to maximize the profit function subject to the constraints below. The method introduced here can be used to solve larger systems that are more complicated.

$$P = 5x + 10y$$
$$8x + 8y \le 160$$
$$4x + 12y \le 180$$
$$x \ge 0;\ y \ge 0$$

Introduce Slack Variables; Rewrite Objective Function

The first step is to rewrite the system without the inequality symbols and to introduce slack variables. We also rewrite the objective function in a form that matches the other equations.

$$8x + 8y + s_1 = 160$$
$$4x + 12y + s_2 = 180$$
$$-5x - 10y + P = 0$$
$$x, y, s_1, s_2 \ge 0$$

Set up the Initial Simplex Tableau

A **simplex tableau** is essentially the same as an augmented matrix from chapter 4. The manipulations on a tableau are also the same as in the Gauss-Jordan method before.

We rename the variable y to x_2, for consistency of notation.

$$\begin{array}{c} \\ s_1 \\ s_2 \\ P \end{array} \begin{array}{c} \begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & P & \end{array} \\ \left(\begin{array}{ccccc|c} 8 & 8 & 1 & 0 & 0 & 160 \\ 4 & 12 & 0 & 1 & 0 & 180 \\ -5 & -10 & 0 & 0 & 1 & 0 \end{array}\right) \end{array}$$

The Initial Simplex Tableau (continued)

$$\begin{array}{rc} & \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & P & \end{array} \\ \begin{array}{r} s_1 \\ s_2 \\ P \end{array} & \left(\begin{array}{ccccc|c} 8 & 8 & 1 & 0 & 0 & 160 \\ 4 & 12 & 0 & 1 & 0 & 180 \\ -5 & -10 & 0 & 0 & 1 & 0 \end{array}\right) \end{array}$$

The variables s_1, s_2, P on the left are the basic variables. A basic variable always has one 1 in its column, otherwise 0. We can read off is value directly from the rightmost column. x_1 and x_2 are 0 (the non-basic variables), so s_1=160, s_2=180, P=0. This is the **initial basic feasible solution** that corresponds to the origin, since $(x_1,x_2) = (0,0)$.

Determine the Pivot Element

$$\begin{array}{rc} & \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & P & \end{array} \\ \begin{array}{r} s_1 \\ s_2 \\ P \end{array} & \left(\begin{array}{ccccc|c} 8 & 8 & 1 & 0 & 0 & 160 \\ 4 & 12 & 0 & 1 & 0 & 180 \\ -5 & -10 & 0 & 0 & 1 & 0 \end{array}\right) \end{array} \quad \begin{array}{l} 160/8 = 20 \\ 180/12 = 15 \\ \end{array}$$

- The most negative element in the last row is –10. Therefore, the x_2-column containing –10 is the **pivot column**.
- To determine the **pivot row**, we divide the coefficients above the -10 into the numbers in the rightmost column and determine the smallest quotient. That happens in the second row, labeled s_2.
- x_2 is the **entering variable**, and s_2 is the **exiting variable**. x_2 will become a basic variable, and s_2 will become non-basic.

The Pivot Step

The goal is to use elementary row operations to produce a 1 in the column of the entering variable (in the row corresponding to the exiting variable), and 0 otherwise.

The elementary row operations are applied the same way as in chapter 4.

Step 1: Divide row s_2 by 12, to get a 1 in pivot position.

$$\begin{array}{rc} & \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & P & \end{array} \\ \begin{array}{r} s_1 \\ s_2 \\ P \end{array} & \left(\begin{array}{ccccc|c} 8 & 8 & 1 & 0 & 0 & 160 \\ 4 & 12 & 0 & 1 & 0 & 180 \\ -5 & -10 & 0 & 0 & 1 & 0 \end{array}\right) \end{array} \longrightarrow \begin{array}{rc} & \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & P & \end{array} \\ \begin{array}{r} s_1 \\ x_2 \\ P \end{array} & \left(\begin{array}{ccccc|c} 8 & 8 & 1 & 0 & 0 & 160 \\ \frac{1}{3} & 1 & 0 & \frac{1}{12} & 0 & 15 \\ -5 & -10 & 0 & 0 & 1 & 0 \end{array}\right) \end{array}$$

The Pivot Step (continued)

Step 2. Obtain zeros in the other positions of the pivot column by subtracting multiples of the pivot row from the other rows.

Relabel the rows: x_2 is the entering variable, and s_2 exits.

$$\begin{array}{rc} & \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & P & \end{array} \\ \begin{array}{r} s_1 \\ x_2 \\ P \end{array} & \left(\begin{array}{ccccc|c} 8 & 8 & 1 & 0 & 0 & 160 \\ \frac{1}{3} & 1 & 0 & \frac{1}{12} & 0 & 15 \\ -5 & -10 & 0 & 0 & 1 & 0 \end{array}\right) \end{array} \longrightarrow \begin{array}{rc} & \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & P & \end{array} \\ \begin{array}{r} s_1 \\ x_2 \\ P \end{array} & \left(\begin{array}{ccccc|c} \frac{16}{3} & 0 & 1 & \frac{-2}{3} & 0 & 40 \\ \frac{1}{3} & 1 & 0 & \frac{1}{12} & 0 & 15 \\ \frac{-5}{3} & 0 & 0 & \frac{5}{6} & 1 & 150 \end{array}\right) \end{array}$$

Find the Next Pivot Element

$$\begin{array}{c|c} & \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & P & \end{array} \\ \begin{array}{c} s_1 \\ x_2 \\ P \end{array} & \left(\begin{array}{ccccc|c} \frac{16}{3} & 0 & 1 & \frac{-2}{3} & 0 & 40 \\ \frac{1}{3} & 1 & 0 & \frac{1}{12} & 0 & 15 \\ \frac{-5}{3} & 0 & 0 & \frac{5}{6} & 1 & 150 \end{array}\right) \end{array}$$

The process must continue since the last row of the matrix still contains a negative value (–5/3).

The next pivot column is column x_1.

Find the Next Pivot Element (continued)

Divide the entries in the last column by the entries in the pivot column, and pick the smallest value. The pivot row is row s_1. The entering variable is x_1, the exiting variable is s_1.

$$\begin{array}{c|c} & \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & P & \end{array} \\ \begin{array}{c} s_1 \\ x_2 \\ P \end{array} & \left(\begin{array}{ccccc|c} \frac{16}{3} & 0 & 1 & \frac{-2}{3} & 0 & 40 \\ \frac{1}{3} & 1 & 0 & \frac{1}{12} & 0 & 15 \\ \frac{-5}{3} & 0 & 0 & \frac{5}{6} & 1 & 150 \end{array}\right) \end{array}$$

$40/(16/3) = 7.5$

$15/(1/3) = 45$

Do the Pivot Step

Step 1: Divide row 1 by 16/3, to get a 1 in pivot position.

$$\begin{array}{c|c} & \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & P & \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ P \end{array} & \left(\begin{array}{ccccc|c} 1 & 0 & \frac{3}{16} & \frac{-1}{8} & 0 & 7.5 \\ \frac{1}{3} & 1 & 0 & \frac{1}{12} & 0 & 15 \\ \frac{-5}{3} & 0 & 0 & \frac{5}{6} & 1 & 150 \end{array}\right) \end{array}$$

Step 2:

- Multiply row 1 by 5/3 and add the result to row 3.
- Multiply row 1 by -1/3 and add result to row 2.

$$\begin{array}{c|c} & \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & P & \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ P \end{array} & \left(\begin{array}{ccccc|c} 1 & 0 & 3/16 & -1/8 & 0 & 7.5 \\ 0 & 1 & -1/16 & 1/8 & 0 & 12.5 \\ 0 & 0 & 5/16 & 5/8 & 1 & 162.5 \end{array}\right) \end{array}$$

Read Off the Solution

Since the last row of the matrix contains no negative numbers, we can stop the procedure. Assign 0 to the non-basic variables and read off $x_1 = x = 7.5$ and $x_2 = y = 12.5$. This is the same solution we obtained geometrically.

The profit is maximized at $162.50

$$\begin{array}{c|c} & \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & P & \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ P \end{array} & \left(\begin{array}{ccccc|c} 1 & 0 & 3/16 & -1/8 & 0 & 7.5 \\ 0 & 1 & -1/16 & 1/8 & 0 & 12.5 \\ 0 & 0 & 5/16 & 5/8 & 1 & 162.5 \end{array}\right) \end{array}$$

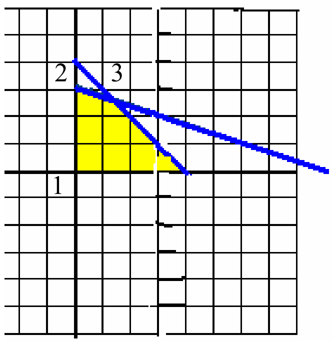
Interpreting the Simplex Process Geometrically

We can now interpret the simplex process just completed geometrically, in terms of the feasible region graphed in the preceding section. The following table lists the three basic feasible solutions we just found using the simplex method, in the order they were found. The table also includes the corresponding corner points of the feasible region.

x_1	x_2	s_1	s_2	P	**Corner Point**
0	0	160	180	0	(0, 0)
0	15	40	0	150	(0, 15)
7.5	12.5	0	0	162.50	(7,5, 12.5)

Interpreting the Simplex Process Geometrically (continued)

Looking at the table on the previous slide, we see that the simplex process started at the origin, moved to the adjacent corner point (0, 15) and then to the optimal solution (7.5, 12.5). This is typical of the simplex process.



Selecting Basic and Nonbasic Variables for the Simplex Method

Given a simplex tableau,

1. *Numbers of variables.* Determine the number of basic variables and the number of nonbasic variables. These numbers do not change during the simplex process.

2. *Selecting basic variables.* A variable can be selected as a basic variable only if it corresponds to a column in the tableau that has exactly one nonzero element (usually 1) and the nonzero element in the column is not in the same row as the nonzero element in the column of another basic variable. This procedure always selects P as a basic variable, since the P column never changes during the simplex process.

Selecting Basic and Nonbasic Variables for the Simplex Method

3. *Selecting nonbasic variables.* After the basic variables are selected in step 2, the remaining variables are selected as the nonbasic variables. The tableau columns under the nonbasic variables usually contain more than one nonzero element.

Selecting the Pivot Element

1. Locate the most negative indicator in the bottom row of the tableau to the left of the P column (the negative number with the largest absolute value). The column containing this element is the *pivot column*. If there is a tie for the most negative indicator, choose either column.
2. Divide each *positive* element in the pivot column above the dashed line into the corresponding element in the last column. The *pivot row* is the row corresponding to the smallest quotient obtained. If there is a tie for the smallest quotient, choose either row. If the pivot column above the dashed line has no positive elements, there is no solution, and we stop.

Selecting the Pivot Element

3. The *pivot* (or *pivot element*) is the element in the intersection of the pivot column and pivot row.

Note: The pivot element is always positive and never appears in the bottom row.

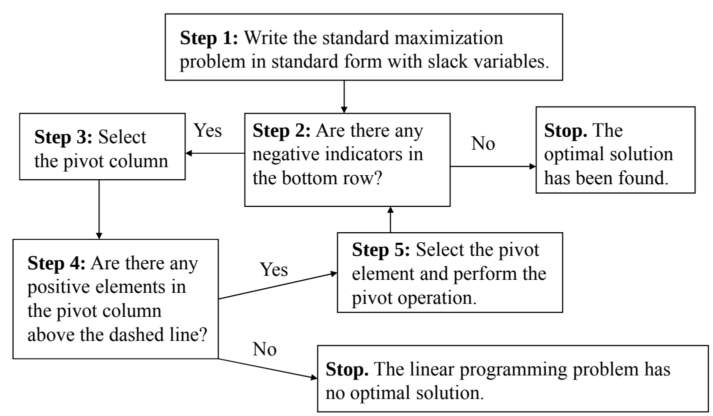
Remember: The entering variable is at the top of the pivot column, and the exiting variable is at the left of the pivot row.

Performing a Pivot Operation

A **pivot operation**, or **pivoting**, consists of performing row operations as follows:

1. Multiply the pivot row by the reciprocal of the pivot element to transform the pivot element into a 1. (If the pivot element is already a 1, omit this step.)
2. Add multiples of the pivot row to other rows in the tableau to transform all other nonzero elements in the pivot column into 0's.

Simplex Method Summarized



Example 2
Agriculture

A farmer owns a 100-acre farm and plans to plant at most three crops. The seed for crops A, B and C costs \$40, \$20, and \$30 per acre, respectively. A maximum of \$3,200 can be spent on seed. Crops A, B, and C require 1, 2, and 1 work days per acre, respectively, and there are a maximum of 160 work days available. If the farmer can make a profit of \$100 per acre on crop A, \$300 per acre on crop B, and \$200 per acre on crop C, how many acres of each crop should be planted to maximize profit?

Example 2
(continued)

The farmer must decide on the number of acres of each crop that should be planted. Thus, the decision variables are

x_1 = number of acres of crop A

x_2 = number of acres of crop B

x_3 = number of acres of crop C

The farmer's objective is to maximize profit:

$$P = 100x_1 + 300x_2 + 200x_3$$

Example 2
(continued)

The farmer is constrained by the number of acres available for planting, the money available for seed, the the available work days. These lead to the following constraints:

$x_1 + x_2 + x_3 \leq 100$ Acreage constraint

$40x_1 + 20x_2 + 30x_3 \leq 3{,}200$ Monetary constraint

$x_1 + 2x_2 + x_3 \leq 160$ Labor constraint

Example 2
(continued)

Adding the nonnegative constraints, we have the following model for a linear programming problem:

Maximize $P = 100x_1 + 300x_2 + 200x_3$

Subject to

$x_1 + x_2 + x_3 \leq 100$

$40x_1 + 20x_2 + 30x_3 \leq 3{,}200$

$x_1 + 2x_2 + x_3 \leq 160$

$x_1, x_2, x_3 \geq 0$

Example 2 (continued)

Next, we introduce slack variables and form the initial system:

$$
\begin{aligned}
x_1 + x_2 + x_3 + s_1 \quad &= 100\\
40x_1 + 20x_2 + 30x_3 \quad + s_2 \quad &= 3200\\
x_1 + 2x_2 + x_3 \quad + s_3 \quad &= 160\\
-100x_1 - 300x_2 - 200x_3 \quad + P &= 0
\end{aligned}
$$

$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$$

The initial system has 7 – 4 = 3 nonbasic variables and 4 basic variables.

Example 2 (continued)

Now we form the simplex tableau and solve by the simplex method:

Enter ↓ (x_2)

$$
\begin{array}{r} \\ s_1 \\ s_2 \\ \text{Exit}\rightarrow s_3 \\ P \end{array}
\begin{array}{c}
\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & P & \end{array}\\
\left[\begin{array}{ccccccc|c}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 100\\
40 & 20 & 30 & 0 & 1 & 0 & 0 & 3,200\\
1 & 2 & 1 & 0 & 0 & 1 & 0 & 160\\
\hline
-100 & -300 & -200 & 0 & 0 & 0 & 1 & 0
\end{array}\right]
\end{array}
\begin{array}{l} \\ \\ \\ 0.5R_3 \text{ -> } R_3 \\ \\ \end{array}
$$

Example 2 (continued)

$$
\begin{array}{r} \\ s_1 \\ s_2 \\ s_3 \\ P \end{array}
\begin{array}{c}
\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & P & \end{array}\\
\left[\begin{array}{ccccccc|c}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 100\\
40 & 20 & 30 & 0 & 1 & 0 & 0 & 3,200\\
0.5 & 1 & 0.5 & 0 & 0 & 0.5 & 0 & 80\\
\hline
-100 & -300 & -200 & 0 & 0 & 0 & 1 & 0
\end{array}\right]
\end{array}
\begin{array}{l} \\ (-1)R_3 + R_1 \text{ -> } R_1 \\ (-20)R_3 + R2 \text{->} R_2 \\ \\ 300R_3 + R_4 \text{->} R_4 \end{array}
$$

Example 2 (continued)

Enter ↓ (x_3)

$$
\begin{array}{r} \\ \text{Exit}\rightarrow s_1 \\ s_2 \\ x_2 \\ P \end{array}
\begin{array}{c}
\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & P & \end{array}\\
\left[\begin{array}{ccccccc|c}
0.5 & 0 & 0.5 & 1 & 0 & -0.5 & 0 & 20\\
30 & 0 & 20 & 0 & 1 & -10 & 0 & 1.600\\
0.5 & 1 & 0.5 & 0 & 0 & 0.5 & 0 & 80\\
\hline
50 & 0 & -50 & 0 & 0 & 150 & 1 & 24,000
\end{array}\right]
\end{array}
\begin{array}{l} \\ 2R_1 \text{ -> } R_1 \\ \\ \\ \\ \end{array}
$$

Example 2 (continued)

$$
\begin{array}{c}
 \\ s_1 \\ s_2 \\ x_2 \\ P
\end{array}
\begin{array}{c}
\begin{array}{ccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & P \end{array} \\
\left[\begin{array}{ccccccc|c}
1 & 0 & 1 & 2 & 0 & -1 & 0 & 40 \\
30 & 0 & 20 & 0 & 1 & -10 & 0 & 1.600 \\
0.5 & 1 & 0.5 & 0 & 0 & 0.5 & 0 & 80 \\
\hline
50 & 0 & -50 & 0 & 0 & 150 & 1 & 24,000
\end{array}\right]
\end{array}
\begin{array}{l}
 \\ \\ (-20)R_1 + R_2 \to R_2 \\ (-0.5)R_1 + R_3 \to R_3 \\ 50R_1 + R_4 \to R_4
\end{array}
$$

Example 2 (continued)

$$
\begin{array}{c}
 \\ x_3 \\ s_2 \\ x_2 \\ P
\end{array}
\begin{array}{c}
\begin{array}{ccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & P \end{array} \\
\left[\begin{array}{ccccccc|c}
1 & 0 & 1 & 2 & 0 & -1 & 0 & 40 \\
10 & 0 & 0 & -40 & 1 & 10 & 0 & 800 \\
0 & 1 & 0 & -1 & 0 & 1 & 0 & 60 \\
\hline
100 & 0 & 0 & 100 & 0 & 100 & 1 & 26,000
\end{array}\right]
\end{array}
$$

All indicators in the bottom row are nonnegative, and we can now read off the optimal solution:

$x_1 = 0$, $x_2 = 60$, $x_3 = 40$, $s_1 = 0$, $s_2 = 800$, $s_3 = 0$, $P = \$26,000$

Example 2 (continued)

Thus, if the farmer plants 60 acres in crop B, 40 acres in crop C and no crop A, the maximum profit of \$26,000 will be realized. The fact that $s_2 = 800$ tells us that this max profit is reached by using only \$2400 of the \$3200 available for seed; that is, we have a slack of \$800 that can be used for some other purpose.