

Chapter 6

Linear Programming: The Simplex Method

Section 1

A Geometric Introduction to the Simplex Method

6.1 Geometric Introduction to the Simplex Method



The geometric method of linear programming from the previous section is limited in that it is only useful for problems involving two decision variables and cannot be used for applications involving three or more decision variables. It is for this reason that a more sophisticated method must be developed.

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George Dantzig 1914 - 2005



George B. Dantzig developed such a method in 1947 while being assigned to the U.S. military. Ideally suited to computer use, the method is used routinely on applied problems involving hundreds and even thousands of variables and problem constraints.

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An Interview with George Dantzig, Inventor of the Simplex Method

http://www.e-optimization.com/directory/trailblazers/dantzig/interview_opt.cfm

IRV

How do you explain optimization to people who haven't heard of it?

GEORGE

I would illustrate the concept using simple examples such as the diet problem or the blending of crude oils to make high-octane gasoline.

IRV

What do you think has held optimization back from becoming more popular?

GEORGE

It is a technical idea that needs to be demonstrated over and over again. We need to show that firms that use it make more money than those who don't.

IRV

Can you recall when optimization started to become used as a word in the field?

GEORGE

From the very beginning of linear programming in 1947, terms like maximizing, minimizing, extremizing, optimizing a linear form and optimizing a linear program were used.

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An Interview with George Dantzig (continued)

GEORGE

The whole idea of objective function, which of course optimization applies, was not known prior to linear programming. In other words, the idea of optimizing something was something that nobody could do, because nobody tried to optimize. So while you are very happy with it and say it's a very familiar term, optimization just meant doing it better than somebody else. And the whole concept of getting the optimum solution just didn't exist. So my introducing the whole idea of optimization in the early days was novel.

IRV

I understand that while programming the war effort in World War II was done on a vast scale, the term optimization as a word was never used. What was used instead?

GEORGE

A program can be thought of as a set of blocks, or activities, of different shapes that can be fitted together according to certain rules, or mass balance constraints. Usually these can be combined in many different ways, some more, some less desirable than other combinations. Before linear programming and the simplex method were invented, it was not possible to computationally determine the best combination such as finding the program that maximizes the number of sorties flown. Instead, all kinds of ground rules were invented deemed by those in charge to be desirable characteristics for a program to have.

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An Interview with George Dantzig (continued)

A typical example of a ground rule that might have been used was: "Go ask General Arnold which alternative he prefers." A big program might contain hundreds of such highly subjective rules. And I said to myself: "Well, we can't work with all these rules." Because what it meant was that you set up a plan. Then you have so many rules that you have to get some resolution of these rules and statements of what they were. To do this, you had to be running to the general, and to his assistants and asking them all kinds of questions.

IRV

Name some of your most important early contributions.

GEORGE

The first was the recognition that most practical planning problems could be reformulated mathematically as finding a solution to a system of linear inequalities. My second contribution was recognizing that the plethora of ground rules could be eliminated and replaced by a general objective function to be optimized. My third contribution was the invention of the simplex method of solution.

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An Interview with George Dantzig (continued)

IRV

And these were great ideas that worked and still do.

GEORGE

Yes, I was very lucky.

IRV

What would you say is the most invalid criticism of optimization?

GEORGE

Saying: "It's a waste of time to optimize because one does not really know what are the exact values of the input data for the program."

IRV

Ok, let's turn this around. What would you say is the greatest potential of optimization?

GEORGE

It has the potential to change the world.

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Standard Maximization Problem in Standard Form

A linear programming problem is said to be a **standard maximization problem in standard form** if its mathematical model is of the following form:

Maximize

$$P = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

subject to problem constraints of the form

$$a_1x_1 + a_2x_2 + \dots + a_nx_n \leq b, \quad b \geq 0$$

with nonnegative constraints $x_1, x_2, \dots, x_n \geq 0$

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Example

We will use a modified form of a previous example. Consider the linear programming problem of maximizing z under the given constraints. This is a standard maximization problem.

$$z = 5x + 10y$$

$$8x + 8y \leq 160$$

$$4x + 12y \leq 180$$

$$x \geq 0; y \geq 0$$

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Example (continued)

There are four lines bordering the feasible region:

$$8x + 8y = 0$$

$$4x + 12y = 0$$

$$x = 0$$

$$y = 0$$

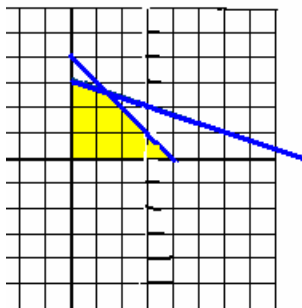
Any two of them intersect in a point. There are a total of six such points.

The coordinates of the point can be found by solving the system of two equations in two unknowns created by the equations of the two lines.

Some of the points are corner points of the feasible region, and some are outside.

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Example (continued)



grid spacing = 5 units

$$z = 5x + 10y$$

$$8x + 8y \leq 160$$

$$4x + 12y \leq 180$$

$$x \geq 0; y \geq 0$$

(0,0)	feasible
(0,15)	feasible
(0,20)	not feasible
(7.5,12.5)	feasible (optimal)
(20,0)	feasible
(45,0)	not feasible

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Slack Variables

To use the simplex method, the constraint inequalities must be converted to equalities. Consider the two constraint inequalities

$$8x + 8y \leq 160$$

$$4x + 12y \leq 180$$

To make this into a system of equations, we introduce **slack variables**

$$8x + 8y + s_1 = 160$$

$$4x + 12y + s_2 = 180$$

They are called slack variables because they take up the slack between the left and right hand sides of the inequalities.

Note that the slack variables must be nonnegative.

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Slack Variables (continued)

- We now have two equations in four unknowns x, y, s_1, s_2
- The system has an infinite number of solutions since there are more unknowns than equations. We can make the system have a unique solution by assigning two of the variables a value of zero and then solving for the remaining two variables. This is called a **basic solution**.
- We select any two of the four variables as **basic variables**. The remaining two variables automatically become **non-basic variables**. The non-basic variables are always assigned a value of zero. Then we solve the equations for the two basic variables.

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Example (continued)

The equations are $8x + 8y + s_1 = 160$

$$4x + 12y + s_2 = 180$$

Select $x = 0$ and $y = 0$ as the non-basic variables. Then the first equation reads $8(0) + 8(0) + s_1 = 160$, so $s_1 = 160$. The second equation becomes $4(0) + 12(0) + s_2 = 180$, so $s_2 = 180$.

This corresponds to the point $(x, y) = (0, 0)$. According to the graph we saw earlier, this is a feasible point.

x	y	s_1	s_2	point	feasible?
0	0	160	180	(0,0)	yes

The other 5 choices of non-basic variables are done similarly. The table on the next slide summarizes the results.

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Example (continued)

Basic Solutions

x	y	s_1	s_2	point	feasible?
0	0	160	180	(0,0)	yes
0	20	0	-60	(0,20)	no
0	15	40	0	(0,15)	yes
20	0	0	100	(20,0)	yes
45	0	-200	0	(45,0)	no
7.5	12.5	0	0	(7.5,12.5)	yes

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Discovery!



- Each basic solution corresponds to an intersection point of the boundary lines of the feasible region.
- A feasible basic solution corresponds to a corner point (vertex) of the feasible region. This includes the optimum solution of the linear programming problem.
- A basic solution that is not feasible includes at least one negative value; a basic feasible solution does not include any negative values.
- We can determine the feasibility of a basic solution simply by examining the signs of all the variables in the solution.

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Generalization

- In a linear programming problem with slack variables there will always be more variables than equations.
- **Basic variables** are selected arbitrarily (as many basic variables as there are equations). The remaining variables are called **non-basic variables**.
- We obtain a **basic solution** by assigning the non-basic variables a value of zero and solving for the basic variables. If a **basic solution** has no negative values, it is **basic feasible solution**.
- **Theorem:** If the optimal value of the objective function in a linear programming problem exists, then that value must occur at one (or more) of the basic feasible solutions.

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Conclusion

This is simply the first step in developing a procedure for solving a more complicated linear programming problem. But it is an important step in that we have been able to identify all the corner points (vertices) of the feasible set without a having three or more variables.

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