**Real Analysis: Chapter 6**

1. Find a function $f(x)$ defined for all $x$and a sequence *{xn}* such that *xn* converges to *4* but *f(xn)* does *not* converge to *f(4)*
2. Use the epsilon-delta definition of the limit of a function to show that $\lim\_{x\to 2}f\left(x\right)=\frac{1}{2} $ for $f\left(x\right)=1/x$.
3. True or false (if true, prove it, if false, give counterexample)
	1. If f is continuous, the *image* of an open set is again open
	2. If f is continuous, the *inverse* image of a connected set is connected
	3. If f is continuous, the *inverse* image of a compact set is compact
	4. A continuous function defined on a compact set is bounded
	5. A continuous function defined on a bounded set is bounded
	6. A continuous function defined on a closed set is bounded
	7. If $|f\left(x\right)|$ is continuous, then $f(x)$ must be continuous
4. Decide (with proof) where, if anywhere, the following functions are continuous (check the text):
	1. f(x) = 1 if x < 0 and f(x) = 0 otherwise
	2. g(x) = 1 if x is rational and g(x) = 0 if x is irrational
	3. h(x) = x if x is rational and h(x) = 0 if x is irrational
	4. k(x) = 1/q if x = p/q is rational and k(x) = 0 if x is irrational *Hint: Use a lemma (with proof) that if* $p\_{n}/q\_{n}$ *converges to any number c, then* $q\_{n}$ *converges to infinity*
5. The function $f(x)=x^{2}$ is continuous on $[0, \infty )$. Is it uniformly continuous on that interval?
6. Consider the functions below. Are they continuous, or do they have a removable discontinuity, a jump discontinuity, or an essential discontinuity at the point where the function splits up:
	1. $f\left(x\right)=\left\{\begin{matrix}\frac{x^{2}-4}{x-2} for x\ne 2\\4 for x=2\end{matrix}\right.$ b) $g\left(x\right)=\left\{\begin{matrix}\frac{x^{2}-4}{x-2} for x\ne 2\\8 for x=2\end{matrix}\right.$
	2. $h\left(x\right)=\left\{\begin{matrix}x sin\left(\frac{1}{x}\right) for x\ne 0\\ 1 for x=0\end{matrix}\right.$ d) $k\left(x\right)=\left\{\begin{matrix}cos\left(\frac{1}{x}\right) for x\ne 0\\ 1 for x=0\end{matrix}\right.$
7. Pick an arbitrary number $c$ and define $f\left(x\right)=|x-c|$ for all $x$. Show that f is uniformly continuous.
8. Show that the function $g\left(x\right)=\sqrt[3]{x^{3n+1}}$ is in $C^{n}(R)$ but not in $C^{n+1}(R)$
9. Find a function that is twice differentiable but the second derivative is not continuous.
10. Define $f\left(x\right)=e^{-1/x^{2} }$ for $x\ne 0$ and $f\left(0\right)=0$. Show that $f^{(n)}\left(x\right)=e^{-1/x^{2}}P(1/x)$ for $x\ne 0$ for all $n$, where $P$ is a polynomial. Use that fact to show that $f$ is a $C^{\infty }$ function and $f^{n}(0)=0$ for all $n$.
11. Prove that $\cos(\left(x\right))=x$ has exactly one solution in the interval $[0, 1]$