**Convergence Tests**

In this section we will list many of the better known tests for convergence or divergence of series.

**Divergence Test**

**If the series http://www.mathcs.org/analysis/reals/symbols/sum%7Ba_n%7D.gif converges, then the sequence http://www.mathcs.org/analysis/reals/symbols/%7Ba_n%7D_n.gif converges to zero. Equivalently:**

**If the sequence http://www.mathcs.org/analysis/reals/symbols/%7Ba_n%7D_n.gif does *not* converge to zero, then the series http://www.mathcs.org/analysis/reals/symbols/sum%7Ba_n%7D.gif cannot converge.**

This test can *never* be used to show that a series *converges,* only to show that a series *diverges*.

## Comparison Test

**Suppose that http://www.mathcs.org/analysis/reals/symbols/sum%7Ba_n%7D.gif converges absolutely, and http://www.mathcs.org/analysis/reals/symbols/%7Bb_n%7D_n.gif is a sequence of numbers for which*| bn | http://www.mathcs.org/analysis/reals/symbols/le.gif | an |* for all *n > N*. Then the series http://www.mathcs.org/analysis/reals/symbols/sum%7Bb_n%7D.gif converges absolutely as well.**

**If the series http://www.mathcs.org/analysis/reals/symbols/sum%7Ba_n%7D.gif converges to positive infinity, and http://www.mathcs.org/analysis/reals/symbols/%7Bb_n%7D_n.gif is a sequence of numbers for which |*an |http://www.mathcs.org/analysis/reals/symbols/le.gif| bn* | for all *n > N*. Then the series http://www.mathcs.org/analysis/reals/symbols/sum%7Bb_n%7D.gif also diverges.**

This is a useful test, but the limit comparison test, which is rather similar, is a much easier to use, and therefore more useful. However, this comparison test is very easy to memorize: Assuming that everything is positive, for simplicity, say we know that:

*| b n | http://www.mathcs.org/analysis/reals/symbols/le.gif | a n |*

for all *n*. then just sum both sides to see what you get formally:

*http://www.mathcs.org/analysis/reals/symbols/sum%7Bb_n%7D.gif http://www.mathcs.org/analysis/reals/symbols/le.gif http://www.mathcs.org/analysis/reals/symbols/sum%7Ba_n%7D.gif*

Then:

* If the left sides equals infinity, so must the right side.
* If the right side is finite, the left side also converges.

**Limit Comparison Test**

**Suppose http://www.mathcs.org/analysis/reals/symbols/sum%7Ba_n%7D.gif and http://www.mathcs.org/analysis/reals/symbols/sum%7Bb_n%7D.gif are two infinite series. Suppose also that**

***r = lim | a n / b n |***

**exists and *0 < r < http://www.mathcs.org/analysis/reals/symbols/infinity.gif* Then http://www.mathcs.org/analysis/reals/symbols/sum%7Ba_n%7D.gif converges absolutely if and only if http://www.mathcs.org/analysis/reals/symbols/sum%7Bb_n%7D.gif converges absolutely.**

This test is more useful than the "direct" comparison test because you do not need to compare the terms of two series too carefully. It is sufficient if the two terms behave similar "in the long run".

**Proof:**

Since *r = lim | a n / b n |* exists, and *r* is between 0 and infinity there exist constants *c* and *C*,*0 < c < C < http://www.mathcs.org/analysis/reals/symbols/infinity.gif* such that for some positive integer *N* we have:

*c < | a n / b n | < C*

if *n > N*. Assume http://www.mathcs.org/analysis/reals/symbols/sum%7Ba_n%7D.gif converges absolutely. From above we have that *c | b n | < | an |*

for *n > N*. Hence, http://www.mathcs.org/analysis/reals/symbols/sum%7Bb_n%7D.gif converges absolutely by the comparison test. Assume http://www.mathcs.org/analysis/reals/symbols/sum%7Bb_n%7D.gif converges absolutely. From above we have that *|a n | < C | b n |*

for *n > N*. But since the series *C http://www.mathcs.org/analysis/reals/symbols/sum%7Bb_n%7D.gif* also converges absolutely, we can use again the comparison test to see that http://www.mathcs.org/analysis/reals/symbols/sum%7Ba_n%7D.gifmust converge absolutely.

## Cauchy Condensation Test

**Suppose http://www.mathcs.org/analysis/reals/symbols/%7Ba_n%7D_n.gif is a decreasing sequence of positive terms. Then the series http://www.mathcs.org/analysis/reals/symbols/sum%7Ba_n%7D.gifconverges if and only if the series http://www.mathcs.org/analysis/reals/numser/graphics/sum_cnd.gif converges.**

This test is rather specialized, just as *Abel's Convergence Test*. The main purpose of the Cauchy Condensation test is to prove that the *p*-series converges if *p > 1*. We will not prove this test.

**Geometric Series**

**Let a be any real number. Then the series http://www.mathcs.org/analysis/reals/symbols/geom(a).gif is called Geometric Series.**

* **if *| a | < 1* the geometric series converges**
* **if *| a | http://www.mathcs.org/analysis/reals/symbols/ge.gif 1* the geometric series diverges**

**If the geometric series converges (i.e. if *| a | < 1*) then *http://www.mathcs.org/analysis/reals/symbols/geom(a).gif = http://www.mathcs.org/analysis/reals/symbols/1fr%7B1ma%7D.gif***

**p Series**

**The series http://www.mathcs.org/analysis/reals/symbols/sum_pser.gif is called a *p* Series.**

* **if *p > 1* the *p*-series converges**
* **if *p http://www.mathcs.org/analysis/reals/symbols/le.gif 1* the *p*-series diverges**

**Proof:**

If *p < 0* then the sequence http://www.mathcs.org/analysis/reals/numser/graphics/pser_1.gif converges to infinity. Hence, the series diverges ([Divergence Test](http://www.mathcs.org/analysis/reals/numser/t_div.html)).

If *p > 0* then consider the series

*http://www.mathcs.org/analysis/reals/numser/graphics/sum_cnd.gif = http://www.mathcs.org/analysis/reals/numser/graphics/pser_2.gif*

The right hand series is now a [Geometric Series](http://www.mathcs.org/analysis/reals/numser/t_geom.html).

* if *0 < p http://www.mathcs.org/analysis/reals/symbols/le.gif 1* then *2 1-p http://www.mathcs.org/analysis/reals/symbols/ge.gif 1*, hence the right-hand series diverges
* if *1 < p* then *2 1-p < 1*, hence the right-hand series converges

Now the result follows from the [Cauchy Condensation test](http://www.mathcs.org/analysis/reals/numser/t_conden.html).

**Root Test**

**Consider the series http://www.mathcs.org/analysis/reals/symbols/sum%7Ba_n%7D.gif. Then:**

* **if *lim sup | a n | 1/n < 1* then the series converges absolutely**
* **if *lim sup | a n | 1/n > 1* then the series diverges**
* **if *lim sup | a n | 1/n = 1*, this test gives no information**

Compare this test with the [Ratio test](http://www.mathcs.org/analysis/reals/numser/t_ratio.html). Although this root test is more difficult to apply, it is better than the ratio test in the following sense: there are series for which the ratio test give no information, yet the root test will be conclusive. You can also use the root test to prove the ratio test, but not visa versa.

The use of the lim sup rather than the regular limit has the advantage that we do not have to be concerned with the existence of a limit. On the other hand, if the regular limit exists, it is the same as the *lim sup*.

**Proof:**

Assume that *lim sup | a n | 1/n < 1*: Because of the properties of the limit superior, we know that there exists *http://www.mathcs.org/analysis/reals/symbols/epsi.gif > 0* and *N > 1* such that

*| a n | 1/n < 1 - http://www.mathcs.org/analysis/reals/symbols/epsi.gif*

for *n > N*. Raising both sides to the *n*-th power we have:

*| a n | < (1 - http://www.mathcs.org/analysis/reals/symbols/epsi.gif) n*

for *n > N*. But the terms on the right hand side form a convergent geometric series. Hence, by the comparison test the series with terms on the left-hand side will converge absolutely. The proof for the second case if left as an exercise.

## Ratio Test

**Consider the series http://www.mathcs.org/analysis/reals/symbols/sum%7Ba_n%7D.gif. Then**

* **if *lim sup | a n+1 / a n | < 1* then the series converges absolutely.**
* **if there exists an *N* such that *| a n+1 / a n | http://www.mathcs.org/analysis/reals/symbols/ge.gif 1* for all *n > N* then the series diverges.**
* **if *lim sup | a n+1 / a n | = 1*, this test gives no information**

**Note that the second condition is true if *lim | an+1 / an |* exists and is strictly bigger than 1.**

The ratio test is easier to use than the Root test . However, there are series for which the ratio tests gives no information, but the root test will. In that sense, the ratio test is weaker than the root test. In addition, the ratio test can be proved using the root test, but not visa versa.

Using the *lim sup* rather than the regular limit has the advantage that we don't have to worry about existence of the limit. However, if the regular limit exists, the *lim sup* yields the same number.

**Proof:**

The proof is very similar to the proof of the root test:

Assume that *lim sup | a n+1 / a n | < 1*: because of the properties of the *lim sup*, we know that there exists *http://www.mathcs.org/analysis/reals/symbols/epsi.gif > 0* and *N > 1* such that

*| a n+1 / a n | < 1 - http://www.mathcs.org/analysis/reals/symbols/epsi.gif*

for *n > N*. Multiplying both sides by |*a n* | we obtain

*| a n+1 | < (1- http://www.mathcs.org/analysis/reals/symbols/epsi.gif) | a n |*

for *n > N*. Therefore, we also have

*| a n+2 | < (1- http://www.mathcs.org/analysis/reals/symbols/epsi.gif) | a n+1 | < (1 - http://www.mathcs.org/analysis/reals/symbols/epsi.gif) 2 | a n |*

for *n > N*. Repeating this procedure, we get, eventually, that

*| a k | < (1 - http://www.mathcs.org/analysis/reals/symbols/epsi.gif) k-N | a N |*

for *k > N*. But the terms on the right hand side form a convergent geometric series, indexed using the variable *k*, where *N*is some constant integer. Hence, by the comparison test the series with terms on the left-hand side will converge absolutely. The proof for the second case if left as an exercise.

## Abel Convergence Test

**Consider the series http://www.mathcs.org/analysis/reals/numser/graphics/abel_1.gif. Suppose that**

1. **the partial sums *S N = http://www.mathcs.org/analysis/reals/symbols/psm%7Ba_n%7D.gif* form a bounded sequence**
2. **the sequence http://www.mathcs.org/analysis/reals/symbols/%7Bb_n%7D_n.gif is decreasing**
3. ***lim b n = 0***

**Then the series http://www.mathcs.org/analysis/reals/numser/graphics/abel_1.gif converges.**

This test is rather sophisticated. Its main application is to prove the Alternating Series test, but one can sometimes use it for other series as well, if the more obvious tests do not work.

## Alternating Series Test

**A series of the form http://www.mathcs.org/analysis/reals/symbols/altser.gif with *b n http://www.mathcs.org/analysis/reals/symbols/ge.gif 0* is called Alternating Series. If the sequencehttp://www.mathcs.org/analysis/reals/symbols/%7Bb_n%7D_n.gif is decreasing and converges to zero, then the sum converges.**

This test does not prove absolute convergence. In fact, when checking for absolute convergence the term 'alternating series' is meaningless.

It is important that the series truly alternates, that is each positive term is followed by a negative one, and visa versa. If that is not the case, the alternating series test does not apply (while Abel's Test may still work).

Let *a n = (-1) n*. Then the formal sum http://www.mathcs.org/analysis/reals/symbols/sum%7Ba_n%7D.gif has bounded partial sums (although the sum does not itself converge). Then, with the given choice of  http://www.mathcs.org/analysis/reals/symbols/%7Bb_n%7D_n.gif  Abel's test applies, showing that the series converges.

## Integral Test

**Suppose that *f(x)* is positive, continuous, decreasing function on the interval *[N, http://www.mathcs.org/analysis/reals/symbols/infinity.gif)*. Let *an = f(n)*. Then**

***http://www.mathcs.org/analysis/reals/symbols/sum%7Ba_n%7D.gif* converges if and only if *http://www.mathcs.org/analysis/reals/numser/graphics/int_1.gif* converges**

Note that this test is much different from all the others. We have not yet formally introduced the concept of an Integral - or even of a continuous function - so that we can not prove this test here. However, for completeness it is included as a test that is sometimes useful to apply.