**Last time: Infinite Series**

**Definition:** The expression *Sn = * is called the **N-th partial sum** of the series, and the sequence { *Sn* } is the sequence of partial sums. If the limit of this sequence *lim Sn* exists and is finite, the series is said to **converge**. If *lim Sn* does not exist or is infinite, the series is said to **diverge**.

**Examples:**

* *= 1 + a + a 2 + a 3 + ...* is called **Geometric Series**. If *| a | < 1* the geometric series converges to **
* ** = 1 + 1/2 + 1/3 + 1/4 + 1/5 + ... is called the **harmonic series**. It diverges (very slowly)
* = 1 - 1/2 + 1/3 - 1/4 + 1/5 + ...   is called **alternating harmonic series**. It converges conditionally.

The proof of convergence for the geometric series consists of a nice trick. Consider the partial sum *S N* and multiply it by *a*:

* *S N = 1 + a + a 2 + a 3 + ... + a N*
* *a S N = a + a 2 + a 3 + ... + a N+1*

Subtracting both equations (of finitely many terms)yields: *(1 - a) SN = 1 - a N+1*. Dividing both sides by *(1 - a)* and taking the limit, the result follows

**Example:** What is the actual limit of the sum ? What is the actual limit of the sum ?

For the harmonic series we need to estimate the *n*-th term in the sequence of partial sums. The *n*-th partial sum for this series is:

*S N = 1 + 1/2 + 1/3 + 1/4 + ... + 1/n*

Now consider the following subsequence extracted from the sequence of partial sums:

*S 1 = 1*

*S 2 = 1 + 1/2*

*S 4 = 1 + 1/2 + (1/3 + 1/4)
       1 + 1/2 + (1/4 + 1/4) = 1 + 1/2 + 1/2 = 1 + 2/2*

*S 8 = 1 + 1/2 + (1/3 + 1/4) + (1/5 + 1/6 + 1/7 + 1/8)
       1 + 1/2 + (1/4 + 1/4) + (1/8 + 1/8 + 1/8 + 1/8) = 1 + 1/2 + 1/2 + 1/2 = 1 + 3/2*

In general, one can use induction (do it as an exercise) to show that

*S 2k  1 + k / 2*

for all *k*. Hence, the subsequence *{ S 2k }* extracted from the sequence of partial sums *{ S N }* is unbounded. But then the sequence *{ S N }* cannot converge either and must diverge to infinity.

Before we can show that the alternating harmonic series converges conditionally, we first need to define:

**Definition (Absolute and Conditional Convergence)**: A series $\sum\_{}^{}a\_{n}$ **converges absolutely** if $\sum\_{}^{}\left|a\_{n}\right|$ converges. The series **converges conditionally** if it converges but not absolutely.

Note: If a series converges absolutely, it converges.

**Examples**:  Does the series  converge absolutely, conditionally, or not at all ? How about the series  ?

### Finally, let’s go back to the alternating harmonic series. It clearly does not converge absolutely (because the series of absolute values is the harmonic series, which diverges). We need to show that it converges “as is”. Here is an elementary proof.

Consider the following two partial sums:

**

and

**

We have that

*S 2n+2 - S 2n = 1 / (2n+1) - 1 / (2n+2) > 0*

and

*S 2n+3 - S 2n+1 = - 1 / (2n+2) + 1/ (2n+3) < 0*

which means for the two subsequences

*{ S 2n }* is monotone increasing and *{ S 2n+1 }* is monotone decreasing

For each sequence we can combine pairs to see that

*S 2n 1* and *S 2n+1 0*

for all *n*. Hence, both subsequences are monotone and bounded and must therefore be convergent. Define their limits as

*lim S 2n = L* and lim S 2n+1 = M

Then

*| M - L | = | lim (S 2n+1 - S 2n) | = 1 / (2n+1)*

which converges to zero. Therefore, *M = L*, i.e. both subsequences converge to the same limit. But this common limit is the same as the limit of the full sequence, because: given any  > 0 we have

* there exists an integer *N* such that *| L - S 2n | < * if *n > N*
* there exists an integer *M* such that *| L - S 2n+1 | < * if *n > M*

Now set *K = max(N, M)*. Then, for the above * > 0* we have

*| L - S n | < *

for *n > K* because *n* is either even or odd. Hence, the alternating harmonic series converges conditionally.

Conditionally convergent series contain a few surprises. As one example, the commutative law no longer applies:

**Theorem: Let****be a conditionally convergent series. Then, for any real number *c* there is a rearrangement of the series such that the new resulting series will converge to *c*.**

As a concrete example, rearrange the alternating harmonic series so that it converges to 2.

In a letter, the mathematician Abel wrote: “On the Whole, [conditionally convergent] series are the Work of the Devil and it's a Shame that one dares base any Demonstration upon them.”