Real Analysis HW 11

1. True or false:
   a. If $\int_a^b f(x)dx = 0$, then $f(x) = 0$ on *[a, b]*
   b. If $\int_x^y f(t)dt = 0$ for all $x, y \in [a, b]$ then $f(x) = 0$ on *[a, b]*
   c. If *f* is continuous on *[a, b]* and $\int_x^y f(t)dt = 0$ for all $x, y \in [a, b]$ then $f(x) = 0$ on *[a, b]*

2. Decide if the following function sequences converge or converge uniformly. *Hint: for some sequences it might help to keep in mind what uniform convergence would imply for the limit function*
   a. $f_n(x) = \frac{x}{n}$ on $[-10,10]$. How about on ***R***?
   b. $f_n(x) = \frac{nx}{1+nx}$ $on\ [0, 1]$.
   c. $f_n(x) = \frac{x^n}{1+x^n}$ on $[0, 1]$. How about on $[0, \infty)$?
   d. $f_n(x) = \frac{x^n}{n+x^n}$ on $[0, 1]$. How about on $[0, \infty)$?
   e. $f_n(x) = nxe^{-nx}$ on [0,2]?. How about on $[2, \infty)$

3. Consider $f_n(x) = \frac{n+\cos(x)}{2n+\sin^2(x)}$. Find the limit function and show that f_n converges uniformly to *f* for all x in ***R***. Then calculate $\lim_{n\to\infty} \int_2^7 f_n(x)\, dx$

4. Show that if f_n are *uniformly* continuous on D and f_n converges uniformly to *f*, then *f* is *uniformly* continuous.

5. Show that if f_n and g_n both converge uniformly to f and g, respectively, then $f_n + g_n$ is uniformly convergent to $f + g$

6. Show that if f_n and g_n both converge uniformly to f and g, respectively, then $f_n\, g_n$ does not necessarily converge uniformly (i.e. find a counter-example).

7. Show that if f_n and g_n both converge uniformly to f and g, respectively, and $|f_n| < M$ and $|g_n| < M$ for all *n* then $f_n\, g_n$ converges uniformly to $f\, g$.

8. How that if $\sum |a_k| < \infty$ then $\sum a_k x^k$ converges uniformly to a continuous function.

9. If $\sum_{n=1}^{\infty} g_n(x)$ is a series of continuous function g_n which converges uniformly on [a, b] to a function *g*. Then

$$\int_a^b g(x)dx = \sum_{n=1}^{\infty} \int_a^b g_n(x)dx$$