# Real Analysis HW 10

1. We know that the Dirichlet function is not Riemann integrable, while the function *f(x) = 1/q if x = p/q is rational and f(x) = 0 if x is not rational* is Riemann integrable and the Riemann integral is zero over the interval [0, 1]. What about

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{else} \end{cases}$$

   Is that function integrable? If so, what is the integral of f over [0, 1]

2. Prove that $f(x) = 3x$ is Riemann integrable over the interval [0, 2] using the **definition** of Riemann integration. Then prove that $f(x) = 3x$ is Riemann integrable over the interval [0, 2] using the **Riemann lemma**.

3. The function $f(x) = 1/x$ has only *one* point of discontinuity on the interval $[-3, 3]$. It should therefore be Riemann integrable on that interval? But *is it* really integrable? If not, explain why not and why this does not violate the theorem that a function that is *almost* continuous is integrable.

4. The function $Si(x)$ is defined via $Si(x) = \int_0^x \frac{\sin(t)}{t}\, dt$. Graph *Si(x)* for *x > 0* by finding critical points, extreme values, inflection points, etc. (if any).

5. Prove that $\lim_{n\to\infty} \int_a^b f(x)\cos(n\,x)\,dx = 0$ for any function $f$ that is continuously differentiable.

6. Suppose *h(x)* and *k(x)* are continuous on [a, b] and $\int_a^b h(x)dx = \int_a^b k(x)dx$. Prove that there exists a number $c \in [a, b]$ such that $h(c) = k(c)$.
   *Hint: Use the Mean Value Theorem 7.2.8 for Integration with $f(x) = h(x) - k(x)$ and $g(x) \equiv 1$.*

7. Find a function $f: [0,1] \to \boldsymbol{R}$ such that $f$ is not Riemann integrable but $|f|$ is Riemann integrable.
   *Hint: Look at a function that is somewhat similar to the Dirichlet function .*

8. Show that if $f$ is Riemann integrable then $f^2$ is also Riemann integrable.
   *Hint: If $|f(x)| \le M$ then $|f^2(x) - g^2(x)| \le 2M|f(x) - g(x)|$ for all $x, y \in [a, b]$. Then estimate $U(f^2, P) - L(f^2, P)$ in terms of $U(f, P) - L(f, P)$ for a given partition.*

9. Find the following integrals:
   a. $\int \frac{2x}{1-x^2}\, dx$
   b. $\int \frac{2}{4-x^2}\, dx$
   c. $\int x^2 \sin(x)\, dx$
   d. $\int e^x \sin(x)\, dx$
   e. $\int \cos^3(x) dx$