Real Analysis HW: Chapter 6

1. Find a function *f(x)* defined for all *x* and a sequence $\{x_n\}$ such that x_n converges to *4* but $f(x_n)$ does *not* converge to *f(4)*.

*According to Prop 6.2.3, continuity preserves limits. That means that if f is continuous at c, and x_n is a sequence converging to c, then $f(x_n)$ would also converge to f(c).Thus, you need a function f that is **not** continuous at 4, a sequence that converges to 4, and then you hope that $f(x_n)$ won't converge to f(4). Perhaps a simple "step" function might do the trick, with a "jump" at 4.*

2. Use the epsilon-delta definition of the limit of a function to show that the limit of *f(x) = 1/x* converges to *½* as *x* converges to *2*. Prove similarly that $\lim_{x\to 2}\frac{1}{x^2}=\frac{1}{4}$, using again the epsilon-delta definition.

You want to show that |f(x) – ½| < epsi as |x – 2| < delta. Similar to what we did with limits of sequences, you'd simplify |f(x) – ½|, and you hope to find the term |x-2| somehow. Then you might be able to guess a delta depending on the epsilon, somehow. Similarly for 1/x^2.

3. Decide (with proof) where, if anywhere, the following functions are continuous:
   a. f(x) = 1 if x < 0 and f(x) = 0 otherwise
   b. g(x) = 1 if x is rational and g(x) = 0 if x is irrational
   c. h(x) = x if x is rational and h(x) = 0 if x is irrational
   d. k(x) = 1/q if x = p/q is rational and k(x) = 0 if x is irrational (for example f(4/9) = 1/9)

The first function is easy, it has a simple jump at 0. Not much to it. You could use a sequence that converges to 0 but $f(x_n)$ does not converge to f(0) to prove discontinuity.

The second function is much more complicated. You could try to take any number x_0. You can find a sequence x_n of rational numbers converging to x_0, as well as a sequence of irrational numbers y_n converging to x_0 (for example if x_0 is rational, $x_n + 1/n$ would be the rational sequence, and $x_n + e/n$ would be an irrational sequence. Now check $f(x_n)$ and $f(y_n)$. If the function was continuous, the limits would have to be the same. Are they? And since x_0 was arbitrary, you have your conclusion.

The third function turns out to be only continuous at x = 0, which you could show by the squeezing theorem and using any sequence converging to 0. You would still have to argue why it is not continuous anywhere else, which you could do similar to the previous function.

*The fourth function is the most complicated. First, for example f(2/13) = 1/13, f(-5/7) = 1/7, f(2) = f(2/1) = 1, and so forth. You will need to use a **lemma** that states that if p_n/q_n converges to any number c, then q_n converges to infinity (hence $1/q_n$ converges to what?). Using this lemma (which you can find, with proof, in section 6.6 - look for the "Countable Discontinuities" function). you could prove that the function is continuous exactly at the irrational points (and not continuous at all rational points). Go imagine that, it's totally crazy.*

4. The function f(x) = x^2 is continuous on [0, infinity). Is it uniformly continuous on that interval? Prove it or give a counter-example.

I would try a counter-example: suppose it was uniformly continuous. Then |f(x) – f(y) | < epsi as long as |x-y| < delta. Pick delta to be one, for example, and pick x = c – 1/3 and y = x + 1/3. Then |x-y| is less than one, no matter what c you pick, but what about |f(x) – f(y)|, especially as c goes to infinity?

5. Suppose the function *f(x)* is continuous at *x*. Show that *|f(x)|* is also continuous at *x*. Is the converse true?

No hints here, you're on your own.

6. Consider the functions below. Are they continuous, or do they have a removable discontinuity, a jump discontinuity, or an essential discontinuity at the point where the function splits up (which is really the only point of interest for each function):

   a. $f(x) = \begin{cases} \frac{x^2-4}{x-2} & for\ x \neq 2 \\ 4 & for\ x = 2 \end{cases}$

   b. $g(x) = \begin{cases} \frac{x^2-4}{x-2} & for\ x \neq 2 \\ 8 & for\ x = 2 \end{cases}$

   c. $h(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & for\ x \neq 0 \\ 1 & for\ x = 0 \end{cases}$

   d. $k(x) = \begin{cases} \cos\left(\frac{1}{x}\right) & for\ x \neq 0 \\ 1 & for\ x = 0 \end{cases}$

   e. $l(x) = \begin{cases} x^2 - 9 & for\ x < 2 \\ x - 3 & for\ x \geq 2 \end{cases}$

   f. $m(x) = \begin{cases} x^2 - 9 & for\ x < 3 \\ x - 3 & for\ x > 3 \end{cases}$

   g. $n(x) = \begin{cases} 3 & for\ x > 0 \\ 4 & for\ x \leq 0 \end{cases}$

Remember, f(x) is continuous at x =c if f(c) exists, the limit as x goes to c exists and that limit is equal to f(c).

A removable discontinuity means that f is really continuous at the point c, even though it does not appear that way at first. For example, for part (a) you notice that the top portion factors and cancels in such a way that you can easily find the limit of f as x goes to 2.

For part (B) it's similar, but f(2) is defined differently, which messes things up. However, if you could redefine f(2) the function would become continuous. Thus, it is …..

For (c) you could use the squeezing theorem to find the limit as x -> 0 of f(x). If that limit matches f(0), it would be continuous, otherwise you might be able to redefine the function at one point to make it continous. Thus, it is ….

For (d) the squeezing theorem won't do anything, but somewhere in chapter 6 there's an example for f(x) = sin(1/x). You should be able to use that result for part (d).

(e) and (f) are related, and since we have not had any jump discontinuity yet, perhaps we have that now.

Finally, the last part is hopefully easy to visualize by just drawing the function.