Real Analysis – Homework 3

Note Title 9/5/2012

① a) Show that $\text{card}((-1,1)) = \text{card}([0,1))$ b) Show $\text{card}((a,b)) = \text{card}([0,1))$

② a) Show that $\text{card}(\mathbb{R}) = \text{card}((0,1))$ b) Show $\text{card}((0,1)) = \text{card}((1,\infty))$

③ Show that $\text{card}((0,1)) = \text{card}([0,1])$ (Wait for a hint on Friday)

④ What is $\text{card}(\mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \ldots)$?

⑤ If S is a finite set, then $\text{card}(S) = \#$ of elements in S. Recall that $P(S)$ is the power set of S, or the set of all subsets of S. If S is finite and $\text{card}(S) = n$, what is $\text{card}(P(S))$? (You don't need to prove it) Note the $\text{card}(P(S)) > \text{card}(S)$ for <u>any</u> set S, finite or infinite (see Thm 2.2.5) – read that proof!

⑥ Let P = set of all polynomials with integer coefficients. Find card(P).

Hint: First define P_n = polynomials of degree n with integer coefficients. Then find card(P_n). Finally, relate P with the P_n's. Use a theorem we proved in class today.

⑦ Recall that a number x_0 is called <u>algebraic</u> if x_0 is a root of a polynomial with integer coefficients. If x_0 is not algebraic, it is called <u>transcendental</u>. For example, all rationals are algebraic, so is $\sqrt{2}$ and $\sqrt[3]{5}$ etc., while π and e are transcendentals. Prove that the algebraic numbers are countable. What about the transcendentals?

8 True or false:

a) if $A \subset B$ and $\mathrm{card}(A) = \mathrm{card}(B)$ then $A = B$

b) if $A \times B$ is finite, then both A and B are finite

c) if $A \times B$ is infinite, then both A and B are infinite

9 Prove (probably by induction):

$$1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{1}{4} n^2 (n+1)^2$$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$ (recursive formula for Binomial coefficients; compare with Pascal's triangle)

Recall: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, $\binom{n}{0} = 1$, $\binom{0}{k} = 0$