

Math 3515 - Final Exam

This is a take-home exam. You may use your notes or the (online) textbook, but you must complete the exam on your own. Please submit the completed exam either in person or via email on the last day of exams.

- Give an example for each of the following. You do **not** have to prove your assertion.
 - A function that is continuous everywhere except at $x = 0$ and $x = 2$ (a graph would suffice here)
 - A function that is not continuous anywhere
 - A function that is continuous everywhere, but not differentiable at $x = 0$ and $x = 2$ (a graph would suffice here)
 - A function that is differentiable everywhere, but whose derivative is not continuous
 - A function that is twice differentiable but not three times differentiable
- Decide whether the following statements are true or false. If false give a counterexample, if true justify your decision.
 - If the integral $\int_a^b f(x)dx = 0$, then $f(x)$ is identically zero on the interval $[a, b]$
 - If a sequence of differentiable functions $\{f_n(x)\}$ converges uniformly to $f(x)$, then the limit function f is continuous.
 - If a sequence of differentiable functions $\{f_n(x)\}$ converges uniformly to $f(x)$, then the limit function f is differentiable.
 - Let $a_n > 0$. If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n^2$ converges.
 - There exists a monotone increasing function that is discontinuous at every irrational number.
 - If f is differentiable on the interval $[a, b]$, then f is uniformly continuous on that interval.
 - The union of a finite collection of compact sets is compact.
- Consider the sequences of functions below. In each case, find the limit function and explain why the convergence cannot be uniform. (Hint: consider the implications of uniform continuity for the limit function, at least in case (a) and (b))
 - $\{f_n(x)\}$, where $f_n(x) = x^n$ for $x \in [0,1]$
 - $\{f_n(x)\}$, where $f_n(x) = e^{-nx}$ for $x \in [0, \infty)$

Extra Credit: $\{f_n(x)\}$, where $f_n(x) = n^2 x^n (1-x)$ for $x \in [0,1]$
- Consider the function $f(x) = \frac{1}{x}$. Is this function uniformly continuous on (a) the interval $[2, 4]$, (b) the interval $[2, \infty)$, or (c) the interval $(0, 2)$? Justify your conclusions.
- Which of the following series is conditionally convergent, absolutely convergent, or divergent? Justify your conclusions.
 - $\sum_{n=1}^{\infty} \frac{(-1)^n n}{6 + n^2}$
 - $\sum_{n=1}^{\infty} \left(\frac{n+1}{n}\right)^{2n}$
 - $\sum_{n=1}^{\infty} n^2 (2c)^n$, $c \in \mathbb{R}$ (Hint: consider several cases; apply the ratio test)
- Each of the functions below has a discontinuity at $x = 0$. Determine whether it is removable, a jump, or essential.
 - $f(x) = \begin{cases} \cos(1/x) & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$
 - $f(x) = \begin{cases} x \cos(1/x) & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$
- Let f be twice continuously differentiable. Prove that if r, s , and t are three roots of the equation $f(x) = 0$ with $r < s < t$, then the equation $f''(x) = 0$ has at least one solution in the interval (r, t) .
- Find the center and the radius of convergence of $\sum_{n=1}^{\infty} \frac{1}{n} (2x - 1)^n$. Also discuss convergence at the endpoints.