Panel 1

## Last Quiz

① Give an example of a sequence of functions $f_n$ s.t. $f_n$ continuous $\forall n$, $f_n$ converges to $f$, but $f$ is not continuous.

② Define "uniform convergence of $\{f_n(x)\}$ to $f(x)$".

③ State the theorem on "Uniform Convergence and Differentiation".

④ State the "Weierstrass Convergence Theorem"
(Extra Credit)

1

Panel 2

$\cancel{f_n(x)} = x/2 \to \frac{x}{2}$

$e^x \to e^x$

$\frac{x^n}{1+x^n} \to \begin{cases} 0 \\ \frac{1}{2} \\ 1 \end{cases}, x \geq 0$

Dirichlet Function

$\sum \frac{1}{x^n} = \sum \left(\frac{1}{x}\right)^n = \frac{1}{1-\frac{1}{x}}, |x|>1$ uniform

$x^n = \begin{cases} 1 & x=1 \\ 0 & x<1 \end{cases}$

$\cancel{\frac{(x+1)}{(x^2-1)}} \to \frac{x+1}{x^2-1}$

$\left(1+\frac{x}{n}\right)^n \to e$

$\frac{1}{n}\sin(n)$

$\left[\left(1-\frac{1}{n}\right)x\right]^n = \left(1-\frac{1}{n}\right)^n x^n \to \frac{1}{e}\cdot 0$ if $|x|<1$

$\frac{1}{e}$ if $x=1$

$x^n/n \to 0$

2

Panel 3

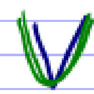
$f_n$ diff'ble + $f_n'$ cont.

(1) $f_n \to f$ pointwise

(2) $f_n' \rightrightarrows g$ uniformly

Then $f$ is diff'ble, $f' = g$

$f_n \rightrightarrows f$, $f_n$ diff'ble but $f$ not $\Rightarrow$



$f_n \to f$, $f_n, f$ diff'ble, $\frac{1}{n}\sin(nx)$

$f_n' \not\rightrightarrows f'$

3

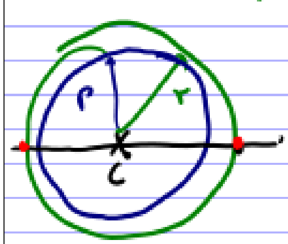
Panel 4

Last Time

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$ power series centered at $c$

radius of convergence: $r =$ use ratio test

converges uniformly and absolutely if $|x-c| \le \rho < r$




need to check endpoints manually

4

Panel 5

Ex: Convergence of $r=1$

a) $\sum \frac{1}{n} x^n$ $\qquad \left| \frac{\frac{1}{n+1} x^{n+1}}{\frac{1}{n} x^n} \right| = \frac{n}{n+1} |x| \xrightarrow{n\to\infty} |x| < 1$

$x=1$: $\sum \frac{1}{n}$ div.

$x=-1$: $\sum \frac{(-1)^n}{n}$ conv. conditionally

b) $\sum \frac{1}{n^2} x^n$ $\qquad \left| \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} \frac{x^{n+1}}{x^n} \right| = \left(\frac{n}{n+1}\right)^2 |x| \to |x| < 1$

$r=1$.

$\left.\begin{array}{l} x=1 \\ x=-1 \end{array}\right\}$ both conv. abs.

5

Panel 6

## Re-Centering

$1+x+x^2$ - recenter at $c=-1$

$a_0 + a_1(x+1) + a_2(x+1)^2 = a_0 + a_1 x + a_1 + a_2 x^2 + 2a_2 x + a_2$

$a_2 = 1$ $\qquad 1+x+x^2 = 1-(x+1)+(x+1)^2$

$a_1 + 2a_2 = 1 \Rightarrow a_1 = -1$

$a_0 + a_1 + a_2 = 1 \Rightarrow a_0 = 1$

$\sum_0^n x^n$ re-center at $c=-1$ $\qquad \sum x^n = \frac{1}{1-x}$ $|x|<1$

$= \sum a_n (x+1)^n$ $\qquad \frac{1}{2-(x+1)} = \frac{1}{2\left(1-\frac{(x+1)}{2}\right)}$

6

Panel 7

$$\sum x^n = \frac{1}{2}\sum\left(\frac{x+1}{2}\right)^n = \sum \frac{1}{2^{n+1}}(x+1)^n$$

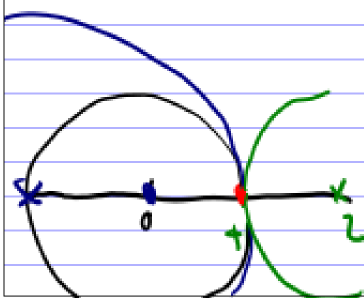
$|x|<1$ $\qquad\qquad$ $|x+1|<2$.

$$x=0 : \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots = 1$$

$$x=0 : \sum_{n=0}^{\infty} \frac{1}{2^n}(x+1)^n = \frac{1}{2} + \frac{1}{4}(x+1) + \frac{1}{8}(x+1)^2 + \dots$$

$$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$$

$$= \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{4} + \dots\right) = \frac{1}{2}\sum_{0}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{2}\frac{1}{1-\frac{1}{2}} =$$




$$= 1$$

7

Panel 8

8

Panel 9

Differentiating and Integrating Power Series

$\sum_{n=0}^{\infty} a_n (x-c)^n$ power series with radius r. Then:

a) limit is cont.

b) $\frac{d}{dx} \sum a_n (x-c)^n = \sum \frac{d}{dx} a_n (x-c)^n = \sum n a_n (x-c)^{n-1}$

c) $\int \sum a_n (x-c)^n = \sum \int a_n (x-c)^n \, dx =$

$= \sum a_n \frac{1}{n+1} (x-c)^{n+1} + C$

9

Panel 10

Ex: Know that $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ $\forall |x| < 1$.

What is $\sum_{n=1}^{\infty} n x^n = x + 2x^2 + 3x^3 + 4x^4 + \ldots$

$= x(1 + 2x + 3x^2 + 4x^3 + \ldots)$

$= x\left(\frac{d}{dx} x + \frac{d}{dx} x^2 + \frac{d}{dx} x^3 + \ldots\right) =$

$= x \frac{d}{dx}(x + x^2 + x^3 + \ldots)$

$= x \frac{d}{dx}\left[x(1 + x + x^2 + \ldots)\right] = x \frac{d}{dx} \frac{x}{1-x}$

$= x \frac{(1-x) + x}{(1-x)^2} = \underline{x / (1-x)^2}$

Panel 11

**Ex:** Know that $\sum_{n=0}^{\infty} x^n =$ $\forall |x|<1$.

What is $\sum_{n=1}^{\infty} n^2 x^n = x + 4x^2 + 9x^3 + 16x^4 + \dots$

$= x\left(1 + 4x + 9x^2 + 16x^3 + \dots\right)$

$= x\left(\frac{d}{dx}x + \frac{d}{dx}2x^2 + \frac{d}{dx}3x^3 + \dots\right) =$

$= x\frac{d}{dx}\left(x + 2x^2 + 3x^3 + \dots\right)$

$= x\frac{d}{dx}\frac{x}{(1-x)^2} = x\frac{(1-x)^2 + x\,2\cancel{(1-x)}}{(1-x)^{3}} =$

$= \frac{x(1+x)}{(1-x)^2}$

Panel 12

$\sum_{n=1}^{\infty} n^3 x^n = ?$

Bonus Points