Panel 1

Power Series

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \ldots + a_n(x-x_0)^n + \ldots$$

is a (formal) power series centered at $x = x_0$.

Ex: $\sum_{n=0}^{\infty} (-1)^n (x+2)^n \qquad a_n = (-1)^n,\ x_0 = -2$

$$\sum_{n=0}^{\infty} (x+2)^{2n} = 1 + (x+2)^2 + (x+2)^4 + \ldots \qquad x_0 = -2$$

$$a_n = \begin{cases} 1 & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd} \end{cases}$$

$$\sum_{n=0}^{\infty} (2x+2)^{2n} = \sum_{0}^{\infty} \left[2(x+1)\right]^{2n}$$

$$= \sum_{0}^{\infty} 2^{2n} (x+1)^{2n}, \quad a_n = \begin{cases} 2^{2n} & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \qquad x_0 = -1$$

Panel 2

$$1 - \tfrac{1}{2}x + \tfrac{1}{4}x^2 - \tfrac{1}{8}x^3 + \ldots = \sum_{n=0}^{\infty} (-1)^n \left(\tfrac{1}{2}\right)^n x^n$$

$$\tfrac{3}{2}x + \tfrac{4}{6}x^2 + \tfrac{5}{24}x^3 + \tfrac{6}{120}x^4 + \ldots = \sum_{n=1}^{\infty} \frac{n+2}{(n+1)!} x^n =$$

$$= \sum_{n=0}^{\infty} \frac{n+3}{(n+2)!} x^{n+1}$$

Thm: Every power series $\sum_{n=0}^{\infty} a_n (x-c)^n$ has properties:

- converges at its center $x = c$
- there is r s.t. series conv. abs. + uniformly for $|x-c| \le \rho$, $\rho < r$, and diverges for $|x-c| > r$.

$r = \lim \left|\frac{a_n}{a_{n+1}}\right|$

r is called "radius of convergence" $(0 \le r \le \infty)$

Panel 3

Ex: $\sum_{n=0}^{\infty} \frac{1}{2^n}(x+2)^n$ find r

Ratio test: $\left|\frac{S_{n+1}}{S_n}\right| = \left|\frac{\frac{1}{2^{n+1}}(x+2)^{n+1}}{\frac{1}{2^n}(x+2)^n}\right| = \frac{1}{2}|x+2| < 1$ want

$\Rightarrow |x+2| < 2 = r$

The proof of thm. was ratio test,

$r = \cancel{\lim \left|\frac{a_n}{a_{n+1}}\right|}$ no good, use Ratio test instead!

$\left|\frac{S_{n+1}}{S_n}\right| = \left|\frac{a_{n+1}}{a_n}\frac{(x-c)^{n+1}}{(x-c)^n}\right| \cdot \left|\frac{a_{n+1}}{a_n}\right||x-c| < 1 \Rightarrow$

$r = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$

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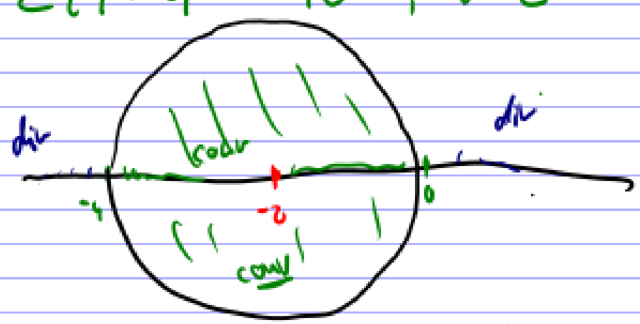
Panel 4

Why Radius of convergence!

$\sum \frac{1}{2^n}|x+2|^n$, $c=-2$, $r=2$ $|x+2| = 2$

$|z+2| = 2$



Ex: $\sum_{n=1}^{\infty} n!(x+3)^n$, $c=-3$. $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!\,(x+3)^{n+1}}{n!\,(x+3)^n}\right| = (n+1)|x+3|$

$\sum_{n=1}^{\infty} \frac{n!}{n^n}(x-4)^n$, $c=4$. $\Rightarrow \lim \left|\frac{a_{n+1}}{a_n}\right| = \infty$ $\forall x \neq -3$

$\Rightarrow r = 1/\infty = 0!$

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Panel 5

$$\sum \frac{n!}{n^n}(x-4)^n \quad : \quad \frac{(n+1)!}{(n+1)^{n+1}}|x-4|^{n+1}\cdot\frac{n^n}{n!\,|x-4|^n} = \frac{\cancel{(n+1)!}}{\cancel{n!}}\frac{n^n}{(n+1)^{n+1}}|x-4|$$

$$= \frac{\cancel{(n+1)}\,n^n}{\cancel{(n+1)}(n+1)^n}|x-4| = \left(\frac{n}{n+1}\right)^n|x-4|$$

Note: $\lim\left(\frac{n}{n+1}\right)^n = \lim \frac{1}{\left(\frac{n+1}{n}\right)^n} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n} = 1/e$

$\lim\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{e}|x-4| \underbrace{<1}_{\text{want!}}$

$\Rightarrow |x-4| < e = r$ radius of conv.!

Panel 6

$\sum (3x-2)^n = \sum 3^n\left(x-\frac{2}{3}\right)^n \qquad c = 2/3,\ r = 1/3$

$\frac{3^{n+1}|x-\frac{2}{3}|^{n+1}}{3^n|x-2/3|^n} = 3|x-\frac{2}{3}| < 1 \qquad r = 1/3$

$\sum \frac{1}{2^n}(x-1)^n \qquad c = 1,\ r = 2 \quad \left(\frac{1/2^{n+1}|x-1|^{n+1}}{1/2^n|x-1|^n} = \frac{1}{2}|x-1| < 1\right)$

Re-center this at $c = 2$: $\sum_{n=0}^{\infty}\frac{1}{2^n}(x-1) = \sum_{n=0}^{\infty} a_n (x-2)^n$?

$\sum_0^{\infty}\frac{1}{2^n}(x-1)^n = \sum_0^{\infty}\left(\frac{x-1}{2}\right)^n = 1/\left(1-\left(\frac{x-1}{2}\right)\right) = \frac{2}{2-(x-1)} = \frac{2}{3-x} = \frac{2}{1-(x-2)}$.

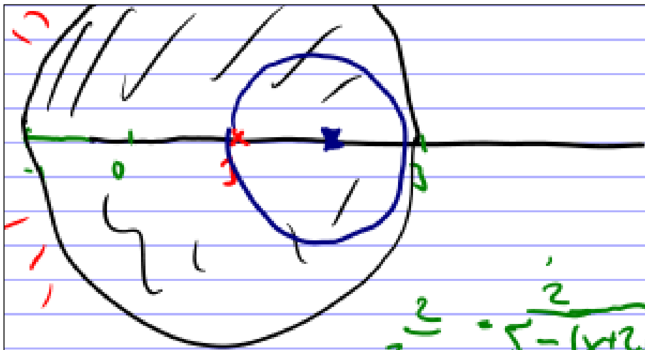
geometric series

conv. if $|x-1| < 2$

$= 2\frac{1}{1-(x-2)} = 2\sum(x-2)^n$

conv. if $|x-2| < 1$

Panel 7



Recenter at $c=-2$

Not possible, or $r=0$

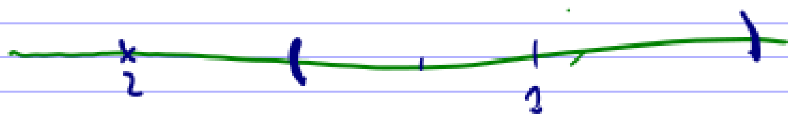
$$\frac{2}{3-x} = \frac{2}{5-(x+2)} = \frac{2}{5\left(1-\left(\frac{x+2}{5}\right)\right)}$$

$$= \frac{2}{5}\sum_{n=0}^{\infty}\left(\frac{x+2}{5}\right)^n \quad \forall\ |x+2|<5$$

Want this $= \sum \frac{1}{2^n}(x-1)^n \quad \forall\ |x-1|<2$

lost

fix this



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Panel 8

Thm: If $\sum a_n(x-c)^n$ is power series, $r>0$. Then

- the series represents a cont. function
- the series is integrable and

$$\int \sum_0^{\infty} a_n(x-c)^n\,dx = \sum_0^{\infty} \int a_n(x-c)^n\,dx = \sum_0^{\infty} \frac{a_n}{n+1}(x-c)^{n+1}$$

- the series is diff'ble and

$$\frac{d}{dx}\sum a_n(x-c)^n = \sum_0^{\infty}\frac{d}{dx}a_n(x-c)^n = \sum_1^{\infty} n a_n(x-c)^{n-1}$$

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Panel 9

Ex: Find $\sum_{n=1}^{\infty} n x^n = x + 2x^2 + 3x^3 + 4x^4 + \dots$

$= x(1 + 2x + 3x^2 + 4x^3 + \dots)$

$= x\left(\frac{d}{dx}x + \frac{d}{dx}x^2 + \frac{d}{dx}x^3 + \dots\right)$

$= x\frac{d}{dx}(x + x^2 + x^3 + \dots) = x \cdot \frac{d}{dx}\frac{}{1-x} = ?$

$= x\frac{d}{dx}(x(1 + x + x^2 + \dots))$

$= x\frac{d}{dx}x \cdot \frac{1}{1-x} = x\frac{d}{dx}\frac{x}{1-x} = x\frac{1-x+x}{(1-x)^2}$

$= \frac{x}{(1-x)^2}$

$\sum n^2 x^n = ?$

$\sum n^3 x^n = ?$

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