Panel 1

Last Time

Uniform convergence: $f_n \rightrightarrows f$ on D iff

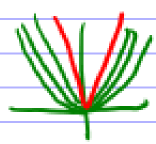
$\|f_n - f\|_D \to 0$

$\Rightarrow$ preserve cont.: f_n cont, $f_n \rightrightarrows f \Rightarrow f$ cont.

$\Rightarrow$ preserve int'ble: f_n int'ble, $f_n \rightrightarrows f \Rightarrow f$ int'ble

Do NOT preserve diff'ble.

$f_n(x) = \frac{1}{n}\sin(nx) \rightrightarrows 0$ but f_n' does not conv.

f_n:  $f = |x|$, f_n diff'ble but f is not

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Panel 2

Neither conv. nor unif. conv. preserves diff'ble.
But as a combo, things work:

Thm: f_n diff'ble on D, f_n' cont on D,
i.e. $f_n \in C^1(D)$. If

$f_n \to f$ (pointwise) on D

$f_n' \rightrightarrows g$ (uniformly) on D

Then f is diff'ble and $f' = g$

[Proof]

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Panel 3

f_n diff'ble, $f_n \to f$

f_n' cont, $f_n' \rightrightarrows g$

unif. cont. preserves cont.

$\Rightarrow g$ is cont.

fund. thm

unif. conv. preserves int.

$$f_n(x) - f_n(a) = \int_a^x f_n'(t)\,dt$$

$$\Rightarrow \lim_{n\to\infty} \int_a^x f_n'(t)\,dt = \int_a^x \lim_{n\to\infty} f_n'(t)\,dt = \int_a^x g(t)\,dt$$

$$\lim_{n\to\infty} f_n(x) - f_n(a) = \underline{f(x)} - f(a) = \int_a^x g(t)\,dt$$

$$f(x) = \int_a^x g(t)\,dt + f(a)$$

$\Rightarrow f'(x) = g(x)$ (Fund. Thm) because g is cont. #

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Panel 4

Sequences of functions
Inf. series
$\Big\} \Rightarrow$ next topic

Def. $f_n(x)$ is a sequence of functions. Define, for each x,

$$\sum_{n=1}^{\infty} f_n(x) = \lim_{N\to\infty} S_N(x)$$

where $S_N(x) = \sum_{n=1}^{N} f_n(x)$

Ex:
$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x|<1$$

$$f_n(x) = x^n, \quad 1 + x + x^2 + x^3 + \dots = \sum x^n = \frac{1}{1-x}$$

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Panel 5

Also:

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n} = \frac{\pi - x}{2}$$

for x with $0 < x < \pi$!

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Panel 6

http://www.mathcs.org/analysis/reals/funseq/answers/funser2.html

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Panel 7

Problem: x^n defined $\forall x$, cont, diff'ble.

But $1 + x + x^2 + x^3 + \dots + x^N + \dots$ is not defined $\forall |x| \geq 1$

sum is more complicated than its parts

Thm: (Weierstrass Convergence Thm)

If f_n is a sequence of functions s.t. $\sum_1^\infty \|f_n\|_D$ converges, then $\sum_1^\infty f_n(x)$ converges abs. + uniformly to some function f. Moreover if f_n are cont., then f is cont.

Ex: $\sum x^n$, $x \in [-r, r]$, $r < 1$. Then $\|x^n\|_{[-r,r]} = r^n$.

$\Rightarrow \sum r^n$ conv. $\Rightarrow \sum x^n$ conv. abs + unif. in $[-r, r]$

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Panel 8

Let's get started with the first step. Fix $x \in D$. Then the series $\sum_{n=0} f(x)$ is a numeric series. In fact, since $|f(x)| \leq \| f \|_D$ that numeric series must converge according to the Comparison Test to a limit L. That limit depends on x so that we can a function

$$F(x) = \sum_{n=0}^{\infty} f(x)$$

which is well-defined (pointwise) for all $x \in D$.

Next we need to show that the convergence is uniform. Take any $\varepsilon > 0$. Since the numeric series $\sum_{n=0}^{\infty} \| f_n \|_D$ converges we can find an integer N such that

$$\sum_{k=n+1}^{\infty} \|f_k\|_D < \varepsilon \text{ for all } n > N$$

But then:

$$|F_n(x) - F(x)| = \left| \sum_{k=n+1}^{\infty} f_k(x) \right| \leq \sum_{k=n+1}^{\infty} |f_k(x)| \leq \sum_{k=n+1}^{\infty} \|f_k\|_D < \varepsilon$$

which means that the sequence of partial sums converges uniformly.

If we knew what *uniformly Cauchy* meant, we could also show that the sequence of partial sums was uniformly Cauchy, which (we would hope) should imply uniform convergence. To practice, define uniformly Cauchy sequences and finish the proof that way.

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Panel 9

Fix $x \in D$. $\sum f_n(x)$ is numeric series, converges by Comp. test $|f_n(x)| \le \|f_n\|_D$. Define limit as

$$F(x) = \sum_{n=1}^{\infty} f_n(x) \quad \forall x \in D \text{ pointwise}$$

Take $\varepsilon > 0$. $\sum \|f_n\| < \infty$ conv.

$$\Rightarrow \sum_{n=k+1}^{\infty} \|f_n\| < \varepsilon \quad \forall k > N$$

$$\Rightarrow |F_N(x) - F(x)| = \left|\sum_{n=N+1}^{\infty} f_n(x)\right| \le \sum_{n=N+1}^{\infty} |f_n(x)| \le \sum_{n=N+1}^{\infty} \|f_n\| < \varepsilon$$

for all x $\Rightarrow F_n \rightrightarrows F$

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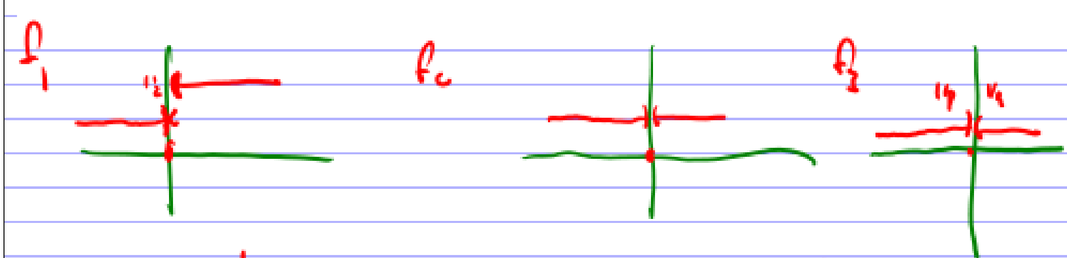
Panel 10

Let $f_n(x) = \begin{cases} \frac{1}{2^n} & if\ x < 0 \\ 0 & if\ x = 0 \\ \frac{1}{n^2} & if\ x > 0 \end{cases}$ Does $\sum_{n=1}^{\infty} f_n(x)$ converge? Is the limit function continuous?



$\|f_n\| \le \frac{1}{n^2}$ $\forall n \ge 8$ and $\sum \|f_n\| \le \sum_{1}^{\infty} \frac{1}{n^2}$ converges

$$\sum_{n=1}^{\infty} f_n \text{ conv. unif + abs. to } f(x) = \begin{cases} \sum_{1}^{\infty} \frac{1}{2^n} = 1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ \sum_{1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, & \text{if } x > 0 \end{cases}$$

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Panel 11

Weierstrass Monster

Ex: $\sum_{n=1}^{\infty} (3/4)^n |\sin(4^n x)|$ is continuous

$f_n(x) = (\frac{3}{4})^n |\sin(4^n x)|$ is cont.,

$\sum \|f_n\| = \sum (3/4)^n < \infty$

Thus $\sum f_n$ conv. abs + unif., and is continuous

Ex: Can you use the Weierstrass convergence theorem for

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$$

No!

How about $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$? Yes!

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Panel 12

Definition 3.3.5: Power Series

A function series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \ldots$$

is called a (formal) *power series* centered at *c*.

other words, a power series is an infinite series of functions where each term consists of a coefficient a_n and a power $(x-$ '. Here are a few examples of power series:

Example 3.3.6: Formal Power Series Examples

- ? All of the following series are power series. List the coefficients a_3 and a_4 for each:

  $$\sum_{n=0}^{\infty} (-1)^n (x+2)^n, \quad \sum_{n=0}^{\infty} (x+2)^{2n}, \quad \sum_{n=0}^{\infty} (2x+2)^{2n}$$

- ? Write the following series in sigma-notation and list the general term a_n:

  - $1 + 2x + 3x^2 + 4x^3 + \ldots$
  - $1 - 1/2\, x + 1/4\, x^2 - 1/8\, x^3 + 1/16\, x^4 \ldots$
  - $3/2\, x + 4/6\, x^2 + 5/24\, x^3 + 6/120\, x^4 + \ldots$

Next time

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