Panel 1

$$\int \cos^3(x)\,dx$$

$$\int \cos(x)\cos^2(x)\,dx$$

$$\int \cos(x)\left[1-\sin^2(x)\right]dx$$

$$\int \cos(x)\,dx - \int \sin^2(x)\cos(x)\,dx$$

$$\sin(x) - \tfrac{1}{3}\sin^3(x)$$

1

Panel 2

If f is R.-able then f^2 is also.

$$U(f^2,P)-L(f^2,P) = \sum_{j=1}^{n}\Big(f^2(\underset{\text{max}}{x_j}) - f^2(\underset{\text{min}}{x_j^*})\Big)\Delta x_j$$

$$= \sum_{j=1}^{n}\big(f(x_j)+f(x_j^*)\big)\big(f(x_j)-f(x_j^*)\big)\Delta x_j$$

$$\le \sum_{j=1}^{n} 2M\big(f(x_j)-f(x_j^*)\big)\Delta x_j =$$

because $|f(x)| \le M$.

$$= 2M\sum_{j=1}^{n}\big(f(x_j)-f(x_j^*)\big)\Delta x_j \le$$

$$\le 2M\big(U(f,P)-L(f,P)\big) \le \varepsilon$$

Take $\varepsilon > 0$. Pick P s.t. $U(f,P)-L(f,P) < \varepsilon/2M$. Then

By Riemann Lemma. f is [illegible]

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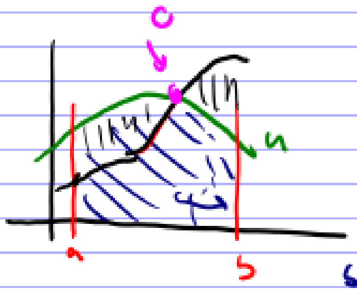
Panel 3

If $\int_a^b h(x)dx = \int_a^b k(x)\,dx$ then $h(c) = k(c)$, $c \in [a,b]$



$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx$$

Take $g(x) \equiv 1$, $f(x) = h(x) - k(x)$.

$$\int_a^b f(x)dx = f(c)(b-a)$$

$$\int_a^b h(x) - k(x)\,dx = \int_a^b h(x)dx - \int_a^b k(x)dx = 0 = (h(c)-k(c))(b-a)$$

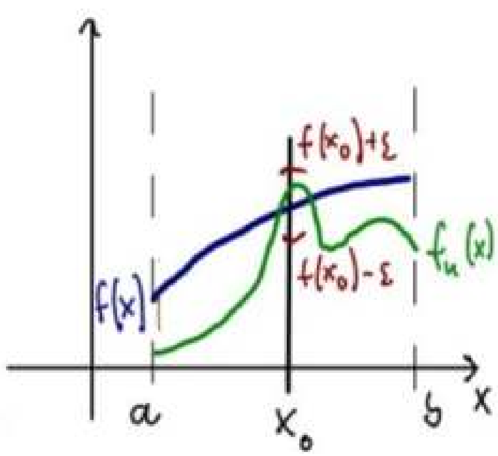
$$\Rightarrow h(c) = k(c).$$

3

Panel 4

Pointwise Convergence

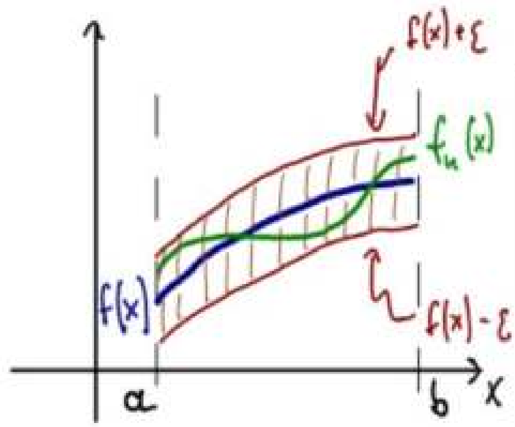
For pointwise convergence we first fix a value x_0. Then we choose an arbitrary neighborhood around $f(x_0)$, which corresponds to a vertical interval centered at $f(x_0)$.



Finally we pick N so that $f_n(x_0)$ intersects the vertical line $x = x_0$ inside the interval $(f(x_0) - \varepsilon, f(x_0) + \varepsilon)$

Uniform Convergence

For uniform convergence we draw an ε-neighborhood around the *entire* limit function f, which results in an "ε-strip" with $f(x)$ in the middle.



Now we pick N so that $f_n(x)$ is completely inside that strip *for all x in the domain.*

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Panel 5

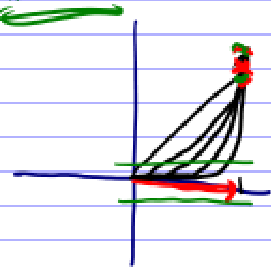
Ex: $f_n(x) = x^n$ on $[0,1]$

$f_n \to f$, $f = \begin{cases} 0 & \text{if } x < 1 \\ 1 & \text{if } x = 1 \end{cases}$ $\quad f_n \overset{?}{\Rightarrow} f$

If $x^n \Rightarrow f$ then $x^n < 1/2 \quad \forall x < 1$

$\Rightarrow \quad x < \sqrt[n]{1/2} = 0.9998$ Not true!

$f_n(x) = x^n$ on $(0,1)$



No, $f_n \cancel{\Rightarrow} f$

same argument

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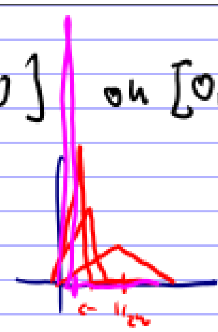
Panel 6

Ex: $f_n(x) = \max\{n - n^2|x - \frac{1}{n}|, 0\}$ on $[0,1]$

$f_n \cancel{\Rightarrow} f = 0$



Recall: f is uniformly cont.

$\Rightarrow f$ is continuous

Clearly: $f_n \Rightarrow f$ implies $f_n \to f$

Also: if f cont. on compact set then f unif. continuous

T/F: If $f_n \to f$ on a compact set, then $f_n \Rightarrow f$

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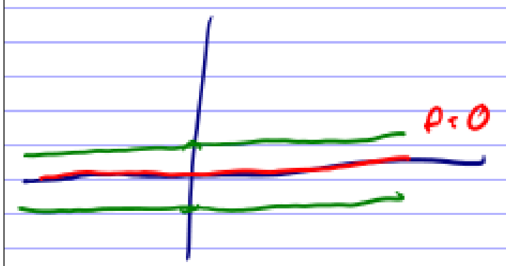
Panel 7

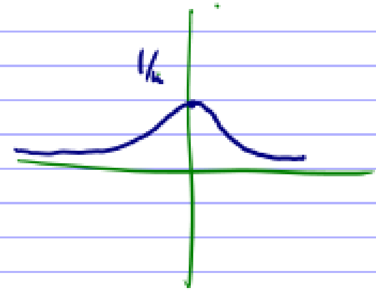
Ex: Find a sequence of functions f_n s.t.

$$f_n \Rightarrow f \quad \forall x \in \mathbb{R}$$

$f_n(x) = \frac{1}{n}\sin(x)$, then $f \Rightarrow 0 \quad \forall x \in \mathbb{R}$

$f_n(x) = \frac{1}{n}\frac{1}{1+x^2} \Rightarrow 0$





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Panel 8

Good news: Uniform convergence is nice!

Thm: f_n cont. on D, $f_n \Rightarrow f$. Then f is cont. on D.

Proof: $|f_n(x) - f(x)| < \varepsilon \quad \forall x \in D$

$|f(x) - f(c)| < \varepsilon$ if $|x-c| < \delta$

$\underline{|f(x) - f(c)|} = |f(x) - f_n(x) + f_n(x) - f_n(c) + f_n(c) - f(c)| \le$

$\le |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)|$

$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$

Take $\varepsilon > 0$. $\exists N$ s.t.

$|f_n(x) - f(x)| < \varepsilon/3 \quad \forall x, n > N$

Pick $\delta > 0$ s.t. $|f_N(x) - f_N(c)| < \frac{\varepsilon}{3}$ if $|x-c| < \delta$

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Panel 9

Def: (Sup-norm) If f is defined on D then

$$\|f\|_D = \sup\{|f(x)| : x \in D\} \quad \to \|\sin(x)\|_{\mathbb{R}} = 1$$

Thm: $f_n \rightrightarrows f$ on D iff

$$\|f_n - f\|_D \to 0$$

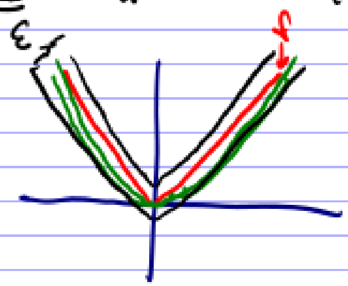
Ex: $f_n(x) = \frac{1}{n}\sin(nx)$. Then $f_n \rightrightarrows 0$ because $\|f_n\|_R = \frac{1}{n} \to 0$

$f_n'(x) = \cos(nx)$ does not converge!

Panel 10

Uniform convergence does not solve every problem

Ex: $f_n \rightrightarrows f$, f_n diff'ble, but f may not be diff'ble



$f_n \rightrightarrows |x|$ f_n diff'ble

$|x|$ is not!

Panel 11

But uniform conv. solve many problems

__Thm:__ f_n cont., $f_n \rightrightarrows f$ on $[a,b]$.

Then: f is intble and $\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx$

i.e. $\lim \int_a^b f_n(x)\,dx = \int_a^b \lim f_n(x)\,dx = \int_a^b f(x)\,dx$

Know: $\|f_n - f\|_{[a,b]} \to 0$

$$\underbrace{\left|\int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx\right|}_{\to 0 \text{ as } n\to\infty} \le \int_a^b |f_n(x) - f(x)|\,dx \le \int_a^b \|f_n - f\|\,dx = (b-a)\|f_n - f\| \to 0$$

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