Panel 1

Last Time: Integration Theorems

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$$\int_a^b f(\underbrace{s(t)}_{u})\,\underbrace{s'(t)\,dt}_{du} = \int_{s(a)}^{s(b)} f(u)\,du$$

$$\int f'(x)\,g(x)\,dx = f\cdot g - \int f\,g'\,dx$$

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Panel 2

$$\int x^2 \cos(x)\,dx = \text{review this (int parts 2x)}$$

$$\int \underbrace{e^x}_{f'} \underbrace{\sin(x)}_{g}\,dx = \text{int parts 2x + solving for integral}$$

$$\int e^x \sin(x)\,dx = \tfrac{1}{2} e^x(\sin(x) - \cos(x))$$

$$e^x \sin(x) - \int \underbrace{e^x}_{f'} \underbrace{\cos(s)}_{g}\,dx =$$

$$e^x \sin(x) - \left[e^x \cos(x) + \int e^x \sin(x)\right]$$

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Panel 3

Pointwise Convergence: $\{f_n(x)\}$ is a sequence of functions s.t. for every fixed x_0 in D the (numeric) sequence $\{f_n(x_0)\}$ converges to $f(x_0)$

$f_n(x) = x^n, \quad x \in [0,1]$

$f(x) = \begin{cases} 1 & \text{if } x=1 \\ 0 & \text{else} \end{cases}$

Take $x_0 = 1 \Rightarrow f_n(x_0) = 1$

Take $x_0 < 1 \Rightarrow (x_0)^n \to 0$

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Panel 4

Bad news about Pointwise Convergence

Ex: If $\{f_n\}$ are cont. and $f_n \to f$ pointwise, then f does not have to be continuous!

Ex: $f_n(x) = x^n, [0,1]$
limit is $f(x) = \begin{cases} 1 & \text{if } x=1 \\ 0 & \text{else} \end{cases}$ not cont.

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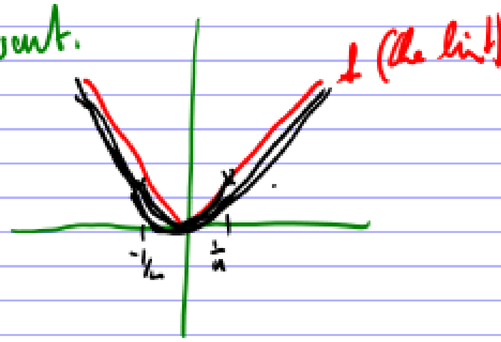
Panel 5

Ex: If $f_n \to f$ pointwise, f_n diffble, then f may not be diffble (but cont.)

$$f_n(x) = \begin{cases} -x - \frac{1}{2n}, & \text{if } -1 \le x \le -\frac{1}{n} \\ \frac{n}{2}x^2, & -\frac{1}{n} < x < \frac{1}{n} \\ x - \frac{1}{2n}, & \frac{1}{n} \le x \le 1 \end{cases}$$



① Show that f is cont at $-1/n$ ② cont at $+1/n$
③ Diffble at $-1/n$ ④ diffble at $+1/n$
⑤ $f_n \to |x|$

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Panel 6

$$f_n(x) = \begin{cases} -x - 1/2n, & \text{if } -1 \le x \le -1/n \\ \frac{n}{2}x^2, & \text{if } -1/n < x < 1/n \\ x - 1/2n, & \text{if } 1/n \le x \le 1 \end{cases}$$

① f_n cont. at $x = -1/n$

$\lim_{x \to -1/n^-} f(x) = -x - \frac{1}{2n}$ $\quad \lim_{x \to -1/n^+} f(x) = \frac{n}{2}x^2 = \frac{n}{2}\left(\frac{-1}{n}\right)^2$

$-\left(\frac{-1}{n}\right) - \frac{1}{2n} = \frac{2n-n}{2n^2} = \frac{1}{2n} = \frac{1}{2n}$ $\quad \frac{n}{2n^2} = \frac{1}{2n}$

② f_n cont. at $x = 1/n$

$\lim_{x \to 1/n^-} f(x) = \lim_{x \to 1/n^-} \frac{n}{2}x^2 = 1/2n$

$\lim_{x \to 1/n^+} f(x) = \lim_{x \to \frac{1}{n}^+} x - \frac{1}{2}n = \lim_{x \to \frac{1}{n}^+} \frac{1}{n} - \frac{1}{2n} = \frac{2-1}{2n} = \frac{1}{2n}$

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Panel 7

$$f_n(x) = \begin{cases} -x - \frac{1}{2n}, & \text{if } -1 \le x \le -\frac{1}{n} \\ \frac{n}{2}x^2, & \text{if } -\frac{1}{n} < x < \frac{1}{n} \\ x - \frac{1}{2n}, & \text{if } \frac{1}{n} \le x \le 1 \end{cases}$$

(3) f_n is diff'ble at $-1/n$

$f_n'(x) = \begin{cases} -1 \\ nx \\ 1 \end{cases} \Rightarrow \lim_{x \to -1/n^+} f_n'(x) = n(-1/n) = -1$ ✓

and $\lim_{x \to -1/n^-} f_n'(x) = -1$

(4) f_n is diff'ble at $x = 1/n$

$$\begin{cases} -1, & -1 \le x \le -1/n \\ nx, & -\frac{1}{n} < x < \frac{1}{n} \\ 1, & 1/n \le x \le 1 \end{cases}$$

$\lim_{x \to \frac{1}{n}^+} 1 = 1$

$\lim_{x \to \frac{1}{n}^-} nx = n\left(\frac{1}{n}\right) = 1$

Panel 8

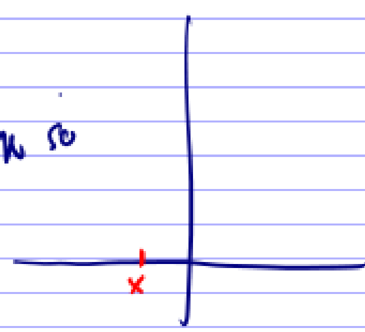
$$f_n(x) = \begin{cases} -x - \frac{1}{2n}, & \text{if } -1 \le x \le -\frac{1}{n} \\ \frac{n}{2}x^2, & \text{if } -\frac{1}{n} < x < \frac{1}{n} \\ x - \frac{1}{2n}, & \text{if } \frac{1}{n} \le x \le 1 \end{cases}$$

(5) $\lim_{n \to \infty} f_n(x) = |x|$

Pick $x = 0 \Rightarrow f_n(0) = 0 \to 0$

$x < 0 \Rightarrow$ make n so large so that $x < -\frac{1}{n} < 0$

$\Rightarrow f_n(x) = -x - \frac{1}{2n} \to -x - 0$



$x > 0$, find n s.t. $0 < \frac{1}{n} < x$

$f_n(x) = x - \frac{1}{2n} \to x$

Panel 9

Ex: Suppose $f_n \to f$ pointwise and all f_n integrable. Then f may not be int'ble

$$f(x) = \begin{cases} 1 & \text{if } x = q_1, q_2, q_3, q_4, \dots \\ 0 & \text{else} \end{cases} \qquad q_j \text{ are } \mathbb{Q} \text{ listed}$$

$$f_n(x) = \begin{cases} 1 & \text{if } x = q_1, q_2, \dots, q_n \\ 0 & \text{else} \end{cases}$$

$f_n \to f$, but f_n is int'ble (finitely many discont.) and f is not!

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Panel 10

Ex: $f_n \to f$ pointwise, f_n diff'ble and f diff'ble. Then $f_n' \not\to f'$

$f_n(x) = \frac{1}{n}\sin(nx)$, f_n is diff'ble then

$f_n'(x) = \cos(nx)$

$\lim_{n\to\infty} f_n(x) = 0$ by squeeze thm. $f(x) = 0$ is diff'ble

But f_n' do not converge! (check $x = \pi$)

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Panel 11

Ex: $f_n \to f$ pointwise, f_n Riemann intble and f R. integrable. Then

$$\lim_{n\to\infty} \int f_n\,dx \neq \int \lim_{n\to\infty} f_n\,dx = \int f\,dx$$

Consider $f_n(x) = \max\{n - n^2|x - \frac{1}{n}|, 0\}$, $x \in [0,1]$

Take $x \in [0, \frac{1}{n}] \Rightarrow n - n^2(\frac{1}{n} - x) = n - n + n^2 x$

$x \in [\frac{1}{n}, \frac{2}{n}] \Rightarrow n - n^2(x - \frac{1}{n}) = 2n - n^2 x$

$x \in [\frac{2}{n}, 1] \Rightarrow n - n^2(x - \frac{1}{n}) \leq n - n^2(\frac{2}{n} - \frac{1}{n}) = 0$

$\Rightarrow f_n(x) = 0$

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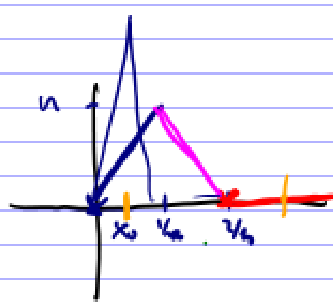
Panel 12

$f_n(x) = \max(n - n^2|x - \frac{1}{n}|, 0) \Rightarrow$

$$f_n(x) = \begin{cases} n^2 x & \text{if } 0 \leq x < \frac{1}{n} \\ 2n - n^2 x & \text{if } \frac{1}{n} < x < \frac{2}{n} \\ 0 & \text{else} \end{cases}$$



$\lim_{n\to\infty} f_n(x) = 0$ because pick $x_0 > 0 \Rightarrow$ find n s.t. $0 < \frac{2}{n} < x$

$\Rightarrow f_n(x) = 0$ if $n > N$

$\Rightarrow \lim_{n\to\infty} f_n(x) = 0$

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Panel 13

$$f_n(x) = \begin{cases} n^2 x & \text{if } 0 \le x < \frac{1}{n} \\ 2n - n^2 x & \text{if } \frac{1}{n} \le x < \frac{2}{n} \\ 0 & \text{else} \end{cases}$$

$$\lim \int_0^1 f_n(x) \neq \int_0^1 \lim f_n(x)dx = 0$$

$$\int_0^1 f_n(x)\,dx = \int_0^{1/n} n^2 x\,dx + \int_{1/n}^{2/n} (2n - n^2 x)\,dx + \int_{2/n}^1 0\,dx$$

$$= n^2 \frac{1}{2} x^2 \Big|_0^{1/n} + 2n x \Big|_{1/n}^{2/n} - n^2 \frac{1}{2} x^2 \Big|_{1/n}^{2/n} =$$

$$= \frac{1}{2} + 2 - \frac{3}{2} = 1 \qquad \text{so done!}$$

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