Panel 1

Def: Suppose f is bdd on $[a,b]$. Define

upper R. integral $\overline{I}^* = \inf\{U(f,P) : P \text{ a partition}\}$.

lower R. integral $\underline{I}_* = \sup\{L(f,P) : P \text{ a partition}\}$

f is integrable if $I^* = I_* = \int_a^b f(x)\,dx$

1

Panel 2

Ex: Show that $f(x) = c$ is integrable.

Any part. P. $\Rightarrow U(f,P) = \sum_{j=1}^{n} c_j (x_j - x_{j-1})$; $c_j = \sup$

$c\sum_{j=1}^{n}(x_j - x_{j-1}) = c(b-a)$

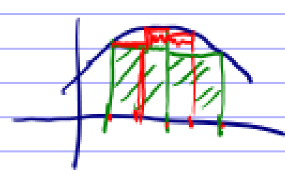
telescoping

$\overline{I}^* = c(b-a)$ Also $\underline{I}_* = c(b-a)$ $\Rightarrow \int_a^b c\,dx = c(b-a)$

Note: If P is a partition, and P' is a refinement of P

$U(f,P) \geq U(f,P')$

$L(f,P) \leq L(f,P')$



2

Panel 3

$f(x) = x^2$ on $[0,1]$ is integrable

/ $|P|$ large won't count for I^*

Proof: Take $\varepsilon > 0$, take partition P with $|P| < \frac{\varepsilon}{2}$. WOLOG

$$|U - L| \leq \sum (d_i - c_i)(x_i - x_{i-1})$$

$d_i = \sup(f \text{ on } [x_{i-1}, x_i])$

$c_i = \inf(f \text{ on } [x_{i-1}, x_i])$

f cont.

$$= \sum (f(t_i) - f(s_i))(x_i - x_{i-1})$$

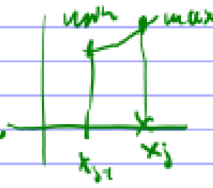
monotone $\checkmark$

$$= \sum \underbrace{(f(x_i) - f(x_{i-1}))}(x_i - x_{i-1})$$

f diff'ble

MVT

$$\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} = f'(a_i)(x_i - x_{i-1}) \leq 2 \frac{\varepsilon}{2} = \varepsilon$$



$$\leq \varepsilon \sum (x_i - x_{i-1}) = \varepsilon(1 - 0) = \varepsilon$$

telescoping

Panel 4

Thm. (Riemann's Lemma)

If f is bdd on $[a,b]$ then f is R-intble iff given $\varepsilon > 0$ there exists one partition P s.t.

$$|U(f,P) - L(f,P)| < \varepsilon$$

Ex: $f(x) = x^2$, $x \in [0,1]$ is integrable.

Pick $P = \{ \frac{j}{n}, j = 0, \dots, n \}$

$$|U - L| \leq \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})|(x_j - x_{j-1}) =$$

$$= \sum_{j=1}^{n} \left(\left(\frac{j}{n}\right)^2 - \left(\frac{j-1}{n}\right)^2 \right) \frac{1}{n} = \frac{1}{n^3} \sum_{j=1}^{n} \cancel{j^2} - (\cancel{j^2} - 2j + 1) =$$

$$\frac{1}{n^3} \left(2 \frac{n(n+1)}{2} - n \right) \xrightarrow[n \to \infty]{} 0$$

Panel 5

Thm: If f is int'ble on $[a,b]$ and P_n is a sequence of partitions with $|P_n| \to 0$, then

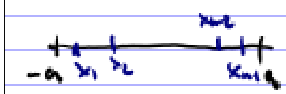
$$\lim_{n\to\infty} R(f, P_n) = \int_a^b f(x)\,dx$$

and f is odd,

Ex: If f is int'ble on $[-a,a]$ then $\int_{-a}^{a} f(x)\,dx = 0$

$P = \{-a + \frac{2j}{n} a\}$ $\quad j = \frac{n}{2} \Rightarrow x_{n/2} = 0$

$x_1 = -a + \frac{2}{n} a \qquad -a + \frac{2(n-1)}{n} a = x_{n-1}$



$-a + \frac{2n}{n} a - \frac{2}{n} a = a - \frac{2}{n} a =$

$= -x_1$

$$\Rightarrow \sum_{j=1}^{n} f(x_j)(x_j - x_{j-1}) = \frac{2}{n}\left(f(x_1) + f(x_2) + \dots + f(x_n)\right) = \frac{2f(a)}{n} \to 0$$

5

Panel 6

Proposition 7.1.12: Properties of the Riemann Integral

Suppose f and g are Riemann integrable functions defined on $[a, b]$. Then

1. $\int_a^b c\, f(x) + d\, g(x)\, dx = c \int_a^b f(x)\, dx + d \int_a^b g(x)\, dx$
2. If $a < c < b$ then $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$
3. $\left| \int_a^b f(x)\, dx \right| \leq \int_a^b | f(x) |\, dx$
4. If g is another function defined on $[a, b]$ such that $g(x) < f(x)$ on $[a, b]$, then $\int_a^b g(x)\, dx \leq \int_a^b f(x)\, dx$
5. If g is another Riemann integrable function on $[a, b]$ then $f(x) \cdot g(x)$ is integrable on $[a, b]$

Proof

6

Panel 7

Thm: If f is continuous on $[a,b]$. Then f is R-integrable. The converse is false.

Proof: f cont. on $[a,b]$ $\Rightarrow$ f is unif. cont.
$\Rightarrow$ given $\varepsilon>0$ $\exists\, \delta$ s.t. $|f(t)-f(s)|<\varepsilon/_{b-a}$ if $|t-s|<\delta$
Take $\varepsilon>0$, pick P with $|P|<\delta$
$\Rightarrow |U-L| \le \sum (d_i-c_i)(x_i-x_{i-1}) \le \sum (f(t_i)-f(s_i))(x_i-x_{i-1})$
$\le \sum \frac{\varepsilon}{b-a}(x_i-x_{i-1}) = \frac{\varepsilon}{b-a}(b-a) = \varepsilon$

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Panel 8

Q: a) Find a function that is not R-integrable
b) Find f that is int'ble but not continuous
c) Find f that is continuous but not diff'ble