Panel 1

Application of Derivatives:

Graph $f(x) = x e^{-x^2}$

$$f' = e^{-x^2} - 2x^3 e^{-x^2} = e^{-x^2}(1-2x^3)$$

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Panel 2

Rolle's Theorem. f cont. on $[a,b]$ and diff'ble on (a,b). If $f(a) = f(b) = 0$ then $\exists c \in (a,b)$ s.t. $f'(c) = 0$

Proof. If $f \equiv 0$ done.

Suppose not. f cont. on cpt set $\Rightarrow$ has max and min. One of them must be not zero (else $f \equiv 0$).

$\Rightarrow$ Assume $f(c) \neq 0$ is max! ($\Rightarrow c \in (a,b)$)   but $f'(c)$ exists so $= 0$

$\Rightarrow$ if $x < c$ then $\frac{f(x)-f(c)}{x-c} < 0 \Rightarrow \lim_{x \to c^-} \frac{f(x)-f(c)}{x-c} \leq 0$

If $x > c$ then $\frac{f(x)-f(c)}{x-c} \geq 0 \Rightarrow \lim_{x \to c^+} \frac{f(x)-f(c)}{x-c} \geq 0$

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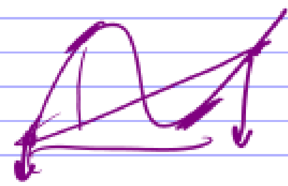
Panel 3

Mean Value Theorem: f is cont. on $[a,b]$ and diff'ble on (a,b). Then $\exists c \in (a,b)$ s.t.

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$



Proof: Want to define a new function $L(x)$ s.t.

$g(x) = f(x) - L(x)$ has property $g(a) = 0 = g(b)$

$\Rightarrow L(a) = f(a)$ and $L(b) = f(b)$ But now

$\Rightarrow L(x) = \frac{f(b)-f(a)}{b-a}(x-a) + f(a)$ $\quad g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} = 0$

qed

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Panel 4

Mean Value Theorem for 2 Functions:

f, g cont. on $[a,b]$, diff'ble on (a,b), and $g'(x) \neq 0$.

Then $\exists c \in (a,b)$ s.t.

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

Proof: Consider $h(x) = f(x) - \frac{f(b)-f(a)}{g(b)-g(a)}[g(x)-g(a)] - f(a)$

$h(a) = 0$, $h(b) = 0$, $h' = f'(c) - \frac{f(b)-f(a)}{g(b)-g(a)} g'(c) = 0$

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Panel 5

Riemann Integration (Chapter 7)

In Calc: think of integrals as area under curve

In Reals: as fancy summation

Def: A partition of $[a,b]$ is a set of points $P = \{x_0, x_1, \dots, x_n\}$ s.t. $a = x_0 < x_1 < x_2 < \dots < x_n = b$

Norm of a partition P = $\max\{|x_j - x_{j-1}|\}$

Refinement of partition P = a part P' that has all points from P as well as some extra point.

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Panel 6

Def: Riemann Sums: Take a partition P of $[a,b]$.

Define n-th Riemann sum of f with respect to P.

$$R(f,P) = \sum_{j=1}^{n} f(t_j)(x_j - x_{j-1}), \quad t_j \in [x_{j-1}, x_j]$$

http://www.mathcs.org/analysis/reals/classes/In…

Panel 7

Ex: Find $R(f,P)$ for $f(x)=x^2$ on $[0,2]$, P a regularly spaced partition of n-points, taking the right endpoint of each subinterval.

$$P=\{0,\tfrac{2}{n},\tfrac{4}{n},\dots\tfrac{2n}{n}=2\}$$

$$R(f,P)=\sum_{j=1}^{n} f\left(\tfrac{2}{n}\cdot j\right)\left(\tfrac{2}{n}j-\tfrac{2}{n}(j-1)\right)=$$

$$=\sum_{j=1}^{n}\left(\tfrac{2}{n}j\right)^2\cdot\tfrac{2}{n}=\tfrac{8}{n^3}\sum_{j=1}^{n} j^2=\tfrac{8}{n^3}\cdot\tfrac{n(n+1)(2n+1)}{6}$$

$$=\tfrac{4}{3}\,\tfrac{n(n+1)(2n+1)}{n^3}\longrightarrow\underline{\tfrac{8}{3}}=\tfrac{x^3}{3}\Big|_0^2=\underline{\tfrac{8}{3}}$$

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Panel 8

Trouble with Riemann sum: $R(f,P)=\sum f(t_j)(x_j-x_{j-1})$
don't know where t_j are.

Solution 1: take left endpoint $\Rightarrow$ left R. Sum
take right endpoint $\Rightarrow$

Problem: no control over size of $f(t_i)$

Solution 2:

Upper S. $U(f,P)=\sum_{j=1}^{n} c_j(x_j-x_{j-1})$, $c_j=\sup(f(x)$ on $[x_{j-1},x_j])$

Lower Sum $L(f,P)=$

Problem: may not be Riemann sums!!

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Panel 9

Q: Are upper / lower sums always Riemann Sums?
If not, find criteria so that they are.

Ex: Find upper/lower sum for Dirichlet function

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Panel 10

Def: Suppose f is bdd on $[a,b]$. Define

$$I^* = \inf\{U(f,P) : P\}$$

$$I_* = \sup\{L(f,P) : P\}$$

I^* is the upper R. Int, I_* is the lower R.I.

Then f is called integrable if $I^* = I_*$

and the common value is called

$$\int_a^b f(x)\,dx$$

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Panel 11

Qu: Is $f(x)=x^2$ integrable on $[0,1]$? Why would it be more difficult on $[-1,1]$?

(it is difficult enough as it is)

http://www.mathcs.org/analysis/reals/integ/ans…

Panel 12

$f(x)=x^2$ on $[0,1]$ is integrable. Find $\int_0^1 f(x)\,dx$!

Proof: Take $\varepsilon > 0$, P a partition with $|P| < \varepsilon/2$

Consider $|U-L| \le \sum |\overset{\sup}{d_j} - c_j|(x_j - x_{j-1})$

f is increasing $\Rightarrow$ $d_j = f(x_j)$ inf $c_j = f(x_{j-1})$

Need to estimate $|d_j - c_j| = |f(x_j) - f(x_{j-1})|$

But $|f(y) - f(x)| \le |f'(c)|\,|y-x| \le 2|y-x|$

$\Rightarrow$ $|f(x_j) - f(x_{j-1})| \le 2|x_j - x_{j-1}| \le 2 \cdot \frac{\varepsilon}{2} = \varepsilon$!

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Panel 13

$\Rightarrow |U - L| \leq \sum |f(x_i) - f(x_{i-1})| \, |x_i - x_{i-1}|$

$\leq \sum_{i=1}^{\infty} \varepsilon |x_i - x_{i-1}| = \varepsilon (x_1 - x_0 + x_2 - x_1 \ldots)$

$= \varepsilon (x_n - x_0) = \varepsilon I = \varepsilon$

Thus: $I^* = I_*$ so $f = x^2$ is integrable

Next: find the value of the integral.

$\Rightarrow$ Pf.: pick ~~[illegible]~~ unif. spaced, helpful

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Panel 14

Is the Dirichlet function intble?

Thm: If f is cont. on $[a,b]$ then f is Riemann intble!

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