Panel 1

Last Time

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Alternatively: If $f(x) = f(c) + M(x-c) + r(x)$

with $\lim_{x \to c} \frac{r(x)}{\|x-c\|} = 0$

then f is diffble at $x=c$, $f'(c) = M$

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Panel 2

Theorem 6.5.7: Algebra with Derivatives

- **Addition Rule**: If f and g are differentiable at x = c then f(x) + g(x) is differentiable at x = c, and
    - $\frac{d}{dx}$ (f(x) + g(x)) = f'(x) + g'(x)
- **Product Rule**: If f and g are differentiable at x = c then f(x) g(x) is differentiable at x = c, and
    - $\frac{d}{dx}$ (f(x) g(x)) = f'(x) g(x) + f(x) g'(x)
- **Quotient Rule**: If f and g are differentiable at x = c, and g(c) # 0 then then f(x) / g(x) is differentiable at x = c, and
    - $\frac{d}{dx}$ (f(x) / g(x)) = $\frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$
- **Chain Rule**: If g is differentiable at x = c, and f is differentiable at x = g(c) then f(g(x)) is differentiable at x = c, and
    - $\frac{d}{dx}$ f(g(x)) = f'(g(x)) g'(x)

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Panel 3

Proof of product rule: $h(x) = f(x)g(x)$

$$\lim_{x\to c} \frac{f(x)g(x) - f(c)g(c)}{x-c} =$$

$$= \lim_{x\to c} \frac{f(x)g(x) - f(x)g(c) + f(x)g(c) - f(c)g(c)}{x-c} =$$

$$= \lim_{x\to c} f(x)\frac{g(x)-g(c)}{x-c} + \lim g(c)\frac{f(x)-f(c)}{x-c} =$$

$$= f(c)\,g'(c) + g(c)\,f'(c).$$

\#

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Panel 4

Ex: $f(x) = x\sin\left(\frac{1}{x}\right)$ has removable discont.
Is the "enhanced" function diffble?

$g(x) = x^2\sin\left(\frac{1}{x}\right)$ also has a removable discont.
Is this "enhanced" function diffble?

T/F: If f is diffble, then f' is continuous?

Let $g(x) = \begin{cases} x^2\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Is g diffble? At $x=0$? $\lim_{h\to 0} \frac{g(0+h)-g(0)}{h} =$

$\lim_{h\to 0} h\sin\left(\frac{1}{h}\right) = 0$

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Panel 5

$$g(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \quad \text{Then}$$

$$g'(x) = \begin{cases} 2x \sin(\frac{1}{x}) + \cancel{x^2} \cos(\frac{1}{x})(-\frac{1}{\cancel{x^2}}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$$= \begin{cases} 2x \sin(\frac{1}{x}) - \cos(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Is g cont. at 0: $\lim_{x \to 0} g(x) = \lim_{x \to 0} 2x \sin(\frac{1}{x}) - \cos(\frac{1}{x})$ d.n.e.

$\Rightarrow g'$ has essential discont

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Panel 6

Ex: Find $f(x)$ s.t. f is n-times diffble and $f^{(n)}$ is cont. but not $(n+1)$-times diffble.

Notation $C^n(U) = \{f : U \to \mathbb{R}$ st. $f^{(n)}$ exists and cont.$\}$

Thus: find $f \in C^n(U)$ but $f \notin C^{n+1}(U)$.

Eg. $n=0$: $f(x) = |x|$ is in C^0 not in C^1!

$\cancel{n=1}$: $f(x) = x^{1/3} \Rightarrow f'(x) = \frac{1}{3x^{2/3}}$ Not if $x=0$

$f \in C^0(\mathbb{R})$ not in $C^1(\mathbb{R})$

$n=1$: $f(x) = x^{4/3} \Rightarrow f'(x) = \frac{4}{3} x^{1/3}$, $f^{(2)}(0)$ d.n.e.

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Panel 7

l'Hospital's Theorem: If f, g are diff'ble at $x=c$

(1) and $f(x) = g(x) = 0$ then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)} \quad \text{(if the limit exists)}$$

(2) and $\lim_{x \to \infty} f(x) = \infty$, $\lim_{x \to \infty} g(x) = \infty$ then

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)} \quad \text{(if the limit exists)}$$

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Panel 8

Ex:

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3} \frac{2x}{1} = 6$$

$$\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{\cos(x)}{1} = 1$$

$$\lim_{x \to \infty} x^4 e^{-x} = \lim_{x \to \infty} \frac{x^4}{e^x} = \lim_{x \to \infty} \frac{4x^{4-1}}{e^x} = \dots = \lim_{x \to \infty} \frac{4!}{e^x} = 0$$

$$\lim_{x \to 0} \frac{1}{x} - \frac{1}{\sin(x)} = \underline{\text{clear!}}$$

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Panel 9

Proof of l'Hospital (part 1)

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f(x)-f(c)^{=0}}{g(x)-g(c)_{=0}}$$

$$= \lim_{x\to c}\frac{\frac{f(x)-f(c)}{x-c}}{\frac{g(x)-g(c)}{x-c}} = \frac{f'(c)}{g'(c)}$$

Not quite good enough! f(W)

Trick: $x_n \nearrow c \Rightarrow \frac{f(x_n)}{g(x_n)} = \frac{\frac{f(x_n)-f(c)}{\cancel{x_n-c}}}{\frac{g(x_n)-g(c)}{\cancel{x_n-c}}} = \frac{f'(c_n)}{g'(d_n)}$ may help.

Panel 10

Ex. Find a function f that diffble $\forall n$, such that $f^{(n)}(0)=0$ $\forall n$, but f is not identically ~~zero~~ constant $(=0)$

Recall If f diffble and $f'(x)=0$ $\forall x \Rightarrow f(x)=\text{const}$

$f(x)=x^n$? $f(0)=0$, $f'(0)=0$, $f''(0)=0, \dots f^{(n)}(0)=n!$

$$f(x)=\begin{cases} e^{-\frac{1}{x^2}} & \text{, if } x\neq 0 \\ 0 & \text{, if } x=0\end{cases}$$

$\lim_{x\to 0} f(x) = \lim_{x\to 0} e^{-1/x^2} = 0$ so cont! $\Rightarrow f\in C^0$

Panel 11

$$f(x) = \begin{cases} e^{-1/x^2} & , x \neq 0 \\ 0 & , x = 0 \end{cases} \qquad f''(x) = \begin{cases} -\frac{6}{x^4} e^{-\frac{1}{x^2}} + \frac{4}{x^6} e^{-1/x^2} \\ 0 \ ? \end{cases}$$

$$f'(x) = \begin{cases} \frac{2}{x^3} e^{-1/x^2} & , \text{if } x \neq 0 \\ & , \text{if } x = 0 \end{cases}$$

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{e^{-1/h^2}}{h} = \lim_{h \to 0} \frac{1/h}{e^{1/h^2}} =$$

$\frac{1}{h} = u$ $$= \lim_{u \to \infty} \frac{u}{e^{u^2}} = \lim_{u \to \infty} \frac{1}{2u e^{u^2}} = 0$$

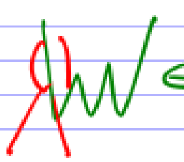
Lemma: $\lim_{h \to 0} \frac{1}{h^n} e^{-1/h^2} = 0$ Hw

Panel 12

"Monster"

Weierstrass Function: f is cont. for all x

but not diffble for any x!

 ← every where you look!

W $\sum \left(\frac{3}{4}\right)^n |\sin(4^n x)|$