Panel 1

Welcome Back - Remember where we left:

Limits. if $|x-c|<\delta \Rightarrow |f(x)-L|<\varepsilon$

if $x_n \to c$ then $f(x_n) \to L$.

Continuity: $\lim_{x\to c} f(x) = f(\lim_{x\to c} x) = f(c)$

Uniform continuous: $|t-s|<\delta \Rightarrow |f(t)-f(s)|<\varepsilon$ $\forall t,s$

Discontinuities — removable

— jump

— essentials

Monotone functions can have at most countable many disc., no essentials

Panel 2

Topology and Continuity

f is continuous iff $f^{-1}(U)$ is open if U is open

$f^{-1}(C)$ is closed if C is closed.

Note: If C is compact $\Rightarrow f^{-1}(C)$

If U is connected $\Rightarrow f^{-1}(U)$

If U is closed: $f^{-1}(U)$ not nec. closed

If U is bdd: $f^{-1}(U)$ not nec. bdd.

Panel 3

Min/Max Theorem: f cont. on compact set $C \Rightarrow f$ has max + min

Proof: $f(C)$ is compact $\Rightarrow \sup\{f(x), x\in C\}$ exist (because sup + inf exists)

$\Rightarrow \sup, \inf \in f(C)$ because its closed

Bolzano's Theorem: If f is cont. on $[a,b]$ and $f(a)\cdot f(b)<0$, i.e. $f(a), f(b)$ have different signs, then $\exists c\in(a,b)$ s.t. $f(c)=0$

Proof: $f(a,b)$ is connected, contains $(f(a), f(b))$. $f(a)\cdot f(b)<0$

$\Rightarrow 0 \in (f(a), f(b)) \Rightarrow \exists c$ s.t. $f(c)=0$

Panel 4

Ex: $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$, n odd

$\Rightarrow p(x)=0$ for at least one solution

$\lim_{x\to\infty} p(x) = \text{sign}(a_n)\infty$, $\lim_{x\to-\infty} p(x) = -\text{sign}(a_n)\infty$

$\Rightarrow \exists x_1$ with $p(x_1)\geq 10$ $\quad \exists x_2$ with $p(x_2) \leq -10$ $\#$

Intermediate Value Thm:

HW

Panel 5

Differentiable Functions:

Def f is diff'ble at $x=c$ if

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c} = f'(c) = \lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$

$x-c=h$

Ex: $f(x)=\sqrt{9-x^2}$. Find $f'(x)=\frac{1}{2}(9-x^2)^{-1/2}\cdot(-2x)$

$$\lim \frac{\sqrt{9-x^2}-\sqrt{9-c^2}}{x-c}\cdot\frac{(\sqrt{\ }+\sqrt{\ })}{(\sqrt{\ }+\sqrt{\ })} = \lim_{x\to c}\frac{(9-x^2)-(9-c^2)}{(x-c)(\sqrt{9-x^2}+\sqrt{9-c^2})} = -(x-c)(x+c)$$

$$\lim_{x\to c}\frac{-(x+c)}{\sqrt{9-x^2}+\sqrt{9-c^2}} = \frac{-2c}{2\sqrt{9-c^2}} = -\frac{c}{\sqrt{9-c^2}}$$

Panel 6

Theorem 6.5.7: Algebra with Derivatives

- **Addition Rule:** If f and g are differentiable at x = c then f(x) + g(x) is differentiable at x = c, and
  - $\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$
- **Product Rule:** If f and g are differentiable at x = c then f(x) g(x) is differentiable at x = c, and
  - $\frac{d}{dx}(f(x)\, g(x)) = f'(x)\, g(x) + f(x)\, g'(x)$
- **Quotient Rule:** If f and g are differentiable at x = c, and g(c) ≠ 0 then then f(x) / g(x) is differentiable at x = c, and
  - $\frac{d}{dx}(f(x) / g(x)) = \frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$
- **Chain Rule:** If g is differentiable at x = c, and f is differentiable at x = g(c) then f(g(x)) is differentiable at x = c, and
  - $\frac{d}{dx} f(g(x)) = f'(g(x))\, g'(x)$

Panel 7

This def. turns out to not be that useful for general settings, e.g. in $\mathbb{R}^2$ or $\mathbb{R}^3$ or $\mathbb{R}^n$.

$\frac{f(x)-f(c)}{x-c}$ ← division does not work!

$$\frac{f(x)-f(c)}{x-c}\approx f'(c) \quad\to\quad f(x)\approx f'(c)(x-c)+f(c)$$

Thm: f defined on (a,b) is diff'ble at $x=c$ if $\exists$ constant M s.t.

$$f(x)=f(c)+M(x-c)+r(x)$$

where $\lim_{x\to c}\frac{r(x)}{\|x-c\|}=0$ Then is $f'=M$

Panel 8

Ex: Show that $f(x)=\sqrt{9+x}$ is diff'ble at $c=0$ using the linear approx. def. $f'(x)=\frac{1}{2}(9+x)^{-1/2}$

$$f(x)\approx f'(c)(x-c)+f(c) = \left(\frac{1}{6}\right)x+3$$

M

$$\sqrt{9+x}=3+\frac{1}{6}x+r(x)$$

$\to r(x)=\sqrt{9+x}-\left(3+\frac{1}{6}x\right)$ Need $\lim_{x\to 0}\frac{r(x)}{x}=0$

$$\frac{\sqrt{9+x}-\left(3+\frac{1}{6}x\right)}{x}\cdot\frac{(\ +\ )}{(\ +\ )} = \frac{9+x-\left(3+\frac{1}{6}x\right)^2}{x\left(\sqrt{9+x}+\left(3+\frac{1}{6}x\right)\right)} = \frac{\cancel{9}+\cancel{x}-\cancel{9}-\cancel{x}-\frac{1}{36}x^2}{\cancel{x}(\sqrt{\ }+(\ ))}$$

$$\to \lim_{x\to 0}\left(\frac{r(x)}{x}\right)=0$$

Panel 9

Thm: f diff'ble at $x=c$ $\Rightarrow$ f is cont at $x=c$

Converse is false ($|x|$) HW

Ex $f(x) = x\sin\left(\frac{1}{x}\right)$ has removable discont.

Is the "enhanced" function diff'ble?

$g(x) = x^2 \sin\left(\frac{1}{x}\right)$ also has a removable discont.

Is this "enhanced" function diff'ble?

T/F: If f is diff'ble, then f' is continuous?

HW

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Panel 10

$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ is cont $\forall x$.

Is it diff'ble. Yes for $x \neq 0$. $f'(x) = \begin{cases} \sin\left(\frac{1}{x}\right) - \frac{1}{x}\cos\left(\frac{1}{x}\right) & x \neq 0 \\ ? & x = 0 \end{cases}$

Is f diff'ble at $x=0$?

$\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = \lim_{h\to 0} \frac{\cancel{h}\sin\left(\frac{1}{h}\right)}{\cancel{h}} = \lim_{h\to 0} \sin\left(\frac{1}{h}\right)$ dne

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