Panel 1

The Story so far..

$\lim_{x \to c} f(x) = L$: given $\varepsilon > 0$ $\exists \delta > 0$ s.t. $|f(x) - L| < \varepsilon$ if $|x - c| < \delta$

$\Leftrightarrow$ for every sequence $x_n \to c$ we are $f(x_n) \to L$

f cont. at $x = c$: given $\varepsilon > 0$ $\exists \delta > 0$ s.t. $|f(x) - f(c)| < \varepsilon$ if $|x - c| < \delta$

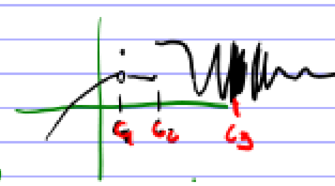
f uniformly cont. in D: given $\varepsilon > 0$ $\exists \delta > 0$ s.t. $|f(s) - f(t)| < \varepsilon$ if $|s - t| < \delta$ $\forall s, t \in D$

1

Panel 2

Theorems: f is uniformly cont. $\Rightarrow$ f cont.

f is cont. on a compact set C $\Rightarrow$ f is unif. cont.



Discontinuity: f not cont. at x_0

removable discontinuity: (c_1) $\lim_{x \to c} f(x)$ exists $(\neq f(c))$

jump discontinuity: (c_2) $\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x)$ exist.

essential discontinuity: else c_3

2

Panel 3

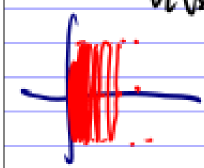
Ex: $f(x) = e^{1/x}$: $\lim_{x \to 0^-} e^{1/x} = 0$ $\lim_{x \to 0^+} e^{1/x} = \infty$

"jump"

squeeze $-x \leq \_ \leq x$

$g(x) = x \sin\left(\frac{1}{x}\right)$: $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$

$h(x) = \cos\left(\frac{1}{x}\right)$: $x_n = \left\{ \frac{2}{(2n+1)\pi} \right\} \Rightarrow \cos\left(\frac{1}{x_n}\right) = \cos\left(\frac{(2n+1)}{2}\pi\right) = 0$



$x_n = \left\{ \frac{1}{(2n+1)\pi} \right\} \Rightarrow \cos\left(\frac{1}{x_n}\right) = \cos((2n+1)\pi) = -1$

$\lim_{x \to 0} h(x)$ d.n.e.

essential.

Panel 4

Monotone Functions and Discontinuities.

Def: f monotone incr.: if $x < y \Rightarrow f(x) < f(y)$

f monotone decr.: if $x < y \Rightarrow f(x) > f(y)$

f is monotone if it is either incr. or decr.

Thm: If f is monotone on $[a,b]$ then:

(1) f can not have essential discont.

(2) There can be at most countably many.

Panel 5

<u>Proof of Theorem</u>: Say f incr. (WOLOG)

(i) $x_n \to x_0$ from left.

$\Rightarrow$ $f(x_n)$ is increasing, bdd by $f(x_0)$ $\to$ converges

$\Rightarrow$ $\lim_{x \to x_0^-} f(x)$ converges.

Similar, $\lim_{x \to x_0^+} f(x)$ conv.

$\Rightarrow$ discont is either jump or removable

(ii) f incr, define $j(c) = \lim_{x \to c^+} f(x) - \lim_{x \to c^-} f(x)$

$j(c) > 0$ because f is increasing. Also

$\sum j(c) \leq f(b) - f(a)$ because f incr.)

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Panel 6

Let $J(n) = \{c : j(c) \geq \frac{1}{n}\}$

$\Rightarrow$ $J(n)$ are finite, because $\sum j(c) < f(b) - f(a)$

$\Rightarrow$ $\bigcup_{n=1}^{\infty} J(n)$ is countable

(countable union of finite sets) q.e.d.

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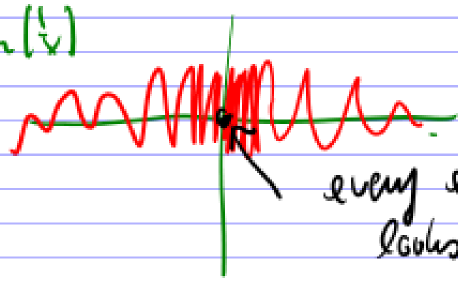
Panel 7

Thm: (How bad are essential discont.)

If f has an essential discont. at $x=c$ then f must change from incr. to decr. in every nbhd. of c.

$\sin\left(\frac{1}{x}\right)$



every essential discont. looks like this!

Panel 8

Continuity and Topology

abs. value

Want def. of continuity that does not depend on $| \; |$

Suppose f is a function with domain $D \subset \mathbb{R}$. Then the following are equivalent:

(1) f is continuous

(2) The inverse image of every (relatively) open set is open

(3) The inverse image of every (relatively) closed set is closed.

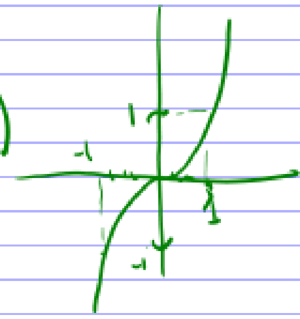
Inverse image $f^{-1}(U) = \{x : f(x) \in U\}$

$U \subset$ range

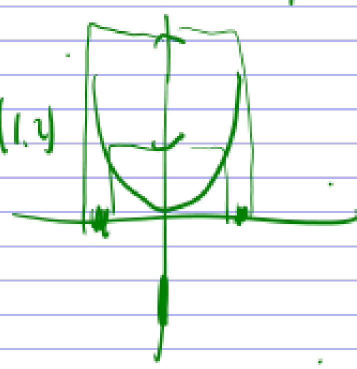
Panel 9

$f(x)=x^3$: $f^{-1}((-1,1)) = (-1,1)$

$f^{-1}((-2,2)) = (-\sqrt[3]{2}, \sqrt[3]{2})$



$f(x)=x^2$:



$f^{-1}((1,4)) = (-2,-1) \cup (1,2)$

$f^{-1}((-1,1)) = (-1,1)$

$f^{-1}((-2,-1)) = \emptyset$

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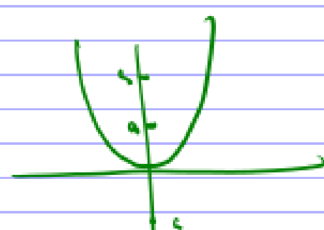
Panel 10

Ex: $f(x)=x^2$ on $\mathbb{R}$ is continuous

Proof. $0<a<s \Rightarrow f^{-1}((a,s)) = (-\sqrt{s},-\sqrt{a}) \cup (\sqrt{a},\sqrt{s})$

open



f^{-1} every open interv. is open $\Rightarrow$ f cont.

$a<0<s$.

$f^{-1}((a,s)) = (-\sqrt{s},\sqrt{s})$ open

$a<s<0$: $f^{-1}((a,s)) = \emptyset$ open

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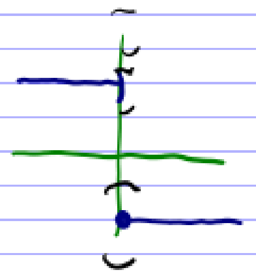
Panel 11

Ex: $f(x) = \begin{cases} 1 & \text{if } x < 0 \\ -1 & \text{if } x \geq 0 \end{cases}$ is not continuous.



$f^{-1}\left(\left(-2, -\frac{1}{8}\right)\right) = [0, \infty)$ closed

(not open)

$\Rightarrow$ not cont.!

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Panel 12

<u>Proof:</u>

12

Panel 13

Q: If f is cont., is the image of every:

(1) open set open? No!

(2) closed set closed? No!

(3) bounded set bounded? No!

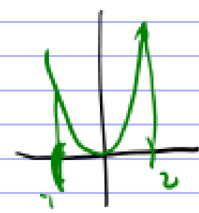
(4) compact set compact? TRUE

(5) connected sets connected? TRUE

(6) disconnected sets disconnected? No!

$[-1,2]$ $[0,4]$

Ex: $f(x) = x^2$ $f((-1,2)) = [0,4)$



$f(A) = \{y \in \text{Range} : y = f(x), x \in A\}$

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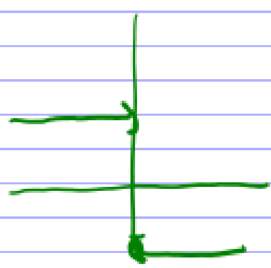
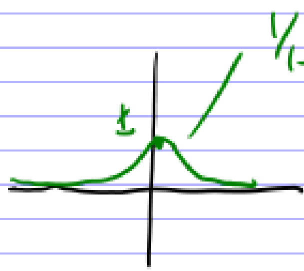
Panel 14

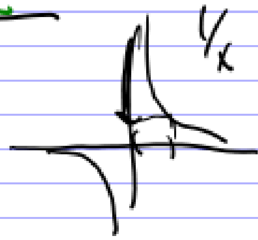


$f([0,\infty)) = \{-1\}$ closed

$f([-1,1]) = \{-1,1\}$ closed



$1/(1+x^2)$ $f([0,\infty)) = (0,1]$ not closed



$1/x$ $f((0,1]) = [1,\infty)$

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Panel 15

f cont., X is connected $\Rightarrow f(X)$ is connected

Assume not, i.e. $f(X) = A \cup B$, A, B open, disjoint (not connected)

$\Rightarrow f^{-1}(A) \cup f^{-1}(B) = X$ check!

open, open + disj. + not (check!)

$\Rightarrow X$ is discon. $\Rightarrow$ ↯

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