Panel 1

Prove that $\lim \frac{3+2n}{2+n} = 2$.

Take $\varepsilon > 0$. Want: N s.t. if $n \geq N$ then

$$\left| \frac{3+2n}{2+n} - 2 \right| < \varepsilon$$

Want

$$\left| \frac{3+2n}{2+n} - \frac{2(2+n)}{2+n} \right| = \left| \frac{-1}{2+n} \right| = \frac{1}{2+n} < \varepsilon$$

$\Rightarrow$ N s.t. $\frac{1}{2+N} < \varepsilon$ $\quad N = \frac{1-2\varepsilon}{\varepsilon}$

$1 < \varepsilon(2+N)$

$2\varepsilon + \varepsilon N$

$$N = \max\left(1, \left\lceil \frac{1-2\varepsilon}{\varepsilon} \right\rceil\right)$$

1

Panel 2

1. Cardinality
2. Sequence (11 or 17)
3. Recursive Sequence
4. 1 or 2 series to check convergence
5. 2 topology questions

2

Panel 3

12: $x_n \to x$. Show that $\frac{x_1+x_2+\dots+x_n}{n} \to x$

$$\left|\frac{x_1+x_2+\dots+x_n}{n} - x\right| \le \frac{|x_1-x|+|x_2-x|+\dots+|x_n-x|}{n}$$

Take any $\varepsilon>0$. $\Rightarrow \exists N_0$ s.t. $|x_j - x| < \varepsilon/2$ if $j > N_0$

$$\frac{|x_1-x|+\dots+|x_{N_0}-x|}{n} + \frac{|x_{N_0+1}-x|+\dots |x_n-x|}{n} < N_0\frac{M}{n} + \frac{(n-N_0)}{n}\frac{\varepsilon}{2}$$

Soln. Pick $N = \max(N_1, N_0)$, N_1 is so large so that $\frac{N_0 \cdot M}{n} < \frac{\varepsilon}{2}$ $\forall n > N_1$

3

Panel 4

What we really want to investigate is cont., deriv., int..

Mix functions, sequences, topology.

Ex: If $x_n \to c$ and f is some function does $f(x_n) \to f(c)$ T/F

i.e. $\lim f(x_n) = f(\lim x_n) = f(c)$

Ex: $f(x) = \begin{cases} 1 & \text{if } x \le 0 \\ 2 & \text{if } x > 0 \end{cases}$, $\{x_n\} = 1/n$.

$x_n \to 0$, $f(x_n) = 2 \ne f(0) = 1$

4

Panel 5

If $\{x_n\}$ div. then $f(x_n)$ diverges ?

$$f(x) = \begin{cases} 1 & \text{if } x \le 0 \\ 2 & \text{if } x > 0 \end{cases} \qquad \{x_n\} = \{3 + (-1)^n\}$$

$$f(x_n) \equiv 2 \to 2$$

If $\{x_n\} = \left\{\frac{(-1)^n}{n}\right\} = \left\{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \ldots\right\}$

$\to f(x_n) = f\left(\frac{(-1)^n}{n}\right) = \left\{f(-1), f\left(\frac{1}{2}\right), f\left(-\frac{1}{3}\right), f\left(\frac{1}{4}\right), \ldots\right\}$

$\{1, 2, 1, 2, 1, 2, 1, 2, \ldots\}$

$$f(x) = \begin{cases} 1 & \text{if } x \le 0 \\ 2 & \text{if } x > 0 \end{cases}$$

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Panel 6

<u>Def.</u> A function f converges to a limit L as x approaches c if: for every sequence $\{x_n\}$ with $x_n \to c$ we have $\{f(x_n)\}$ converges to L.

<u>Ex:</u> $f(x) = 2x + 5$. Prove that $\lim_{x \to 2} f(x) = 9$

Take any $x_n \to 2$. Then

$\Rightarrow 2x_n \to 4$

$\Rightarrow 2x_n + 5 \to 4 + 5 = 9$

6

Panel 7

Def (Alternate) A function f converges to L as x goes to c, if for every $\varepsilon > 0$ there is a $\delta > 0$ s.t.

$$|f(x) - L| < \varepsilon$$

whenever $|x - c| < \delta$

Thm Both def's are equivalent. HW

Ex: $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{else} \end{cases}$ (Dirichlet function)

No graph?

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Panel 8

What is $\lim_{x \to c} f(x)$ d.n.e. for any c!

$c = 1$: $\lim_{x \to 1} f(x)$ d.n.e.

take $x_n \in \mathbb{Q}$, $x_n \to 1$ (e.g. $x_n = 1 + \frac{1}{n}$) $f(x_n) = 1$

$x_n \notin \mathbb{Q}$, $x_n \to 1$ (e.g. $x_n = 1 + \frac{\sqrt{2}}{n}$) $f(x_n) = 0$

$f = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{else} \end{cases}$

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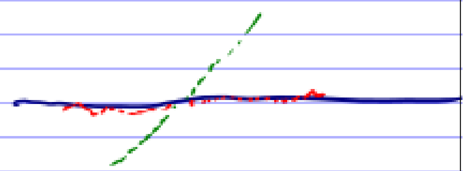
Panel 9

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{else} \end{cases}$$



What is $\lim_{x \to 0} f(x) = 0$

Proof: take any $\{x_n\} \to 0$

$$\Rightarrow |f(x_n)| \le |x_n| \to 0$$

$$\Rightarrow f(x_n) \to 0$$

$\lim_{x \to c \neq 0} f(x) =$ undef.!

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Panel 10

$$f(x) = \begin{cases} 1/q & \text{if } x = p/q \in \mathbb{Q} \\ 0 & \text{else} \end{cases}$$

$f(\frac{11}{13}) = \frac{1}{13}$, $f(\frac{1}{2}) = \frac{1}{2}$, ... $f(5) = 1$, $f(\pi) = 0$

Think about the "graph" of this

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