Panel 1

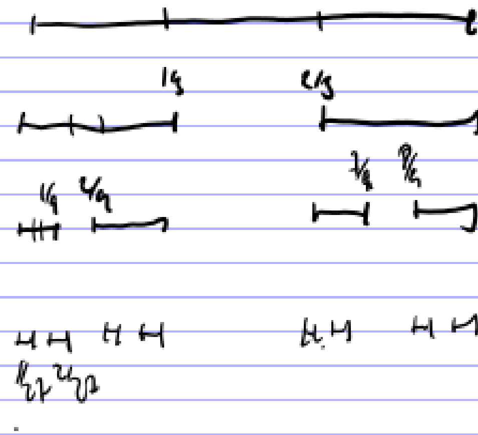
# The Cantor Set

Let $C_1 = [0,1]$

Let $C_2 = C_1 - \left(\frac{1}{3}, \frac{2}{3}\right)$

$C_3 = C_2 - \left(\frac{1}{9}, \frac{2}{9}\right) - \left(\frac{7}{9}, \frac{8}{9}\right)$

$C_4 = C_3 - \left(\frac{1}{27}, \frac{2}{27}\right) - \left(\frac{7}{27}, \frac{8}{27}\right) - \left(\frac{19}{27}, \frac{20}{27}\right) - \left(\frac{25}{27}, \frac{26}{27}\right)$




$\vdots$

$C = \bigcap_{n=1}^{\infty} C_n$ "$= \lim C_n$"

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Panel 2

Define the Cantor set: $C = \bigcap_{n=1}^{\infty} C_1$

① The Cantor set is not empty: $0, 1 \in C$ ✓

② How many elements are in $C$? ↗ infinite

$\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots \in C$; in fact all endpoints of little sets.

④ Are the following numbers in $C$?

a) $1/4$

b) $8/27 \in C$ because it is endpoint!

c) $31/81$ $\quad \frac{1}{3} = \frac{27}{81} < \frac{31}{81} < \frac{54}{81} = \frac{2}{3}$ $\quad \frac{31}{81} \notin C$

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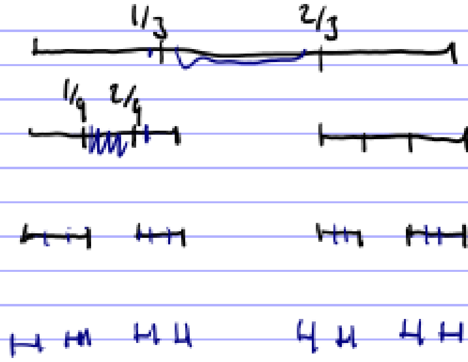
Panel 3

$C_1 = [0,1]$

$C_2 = C_1 - \left(\frac{1}{3}, \frac{2}{3}\right)$

$C_3 = C_2 - \left(\frac{1}{9}, \frac{2}{9}\right) - \left(\frac{7}{9}, \frac{8}{9}\right)$

$C_4 = C_3 - \left(\frac{1}{27}, \frac{2}{27}\right) - \left(\frac{7}{27}, \frac{8}{27}\right) - \left(\frac{17}{27}, \frac{19}{27}\right) - \left(\frac{25}{27}, \frac{26}{27}\right)$




$\frac{1}{4} < \frac{1}{3}$ stays in $C_2$

$\frac{2}{9} = 0.22 < \frac{1}{4} < \frac{3}{9}$ stays in $C_3$

$\frac{6}{27} < \frac{1}{4} < \frac{7}{27}$ stays in $C_4$

looks like $\frac{1}{4} \in C$

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Panel 4

(5) $C$ is compact.

$C \subset [0,1]$ so bdd.

$C_2 = C_1 - \left(\frac{1}{3}, \frac{2}{3}\right)$

$C_3 = C_2 - \left[\left(\frac{1}{9}, \frac{2}{9}\right) \cup \overbrace{\left(\frac{4}{9}, \frac{5}{9}\right)}^{\text{not in } C_2 \text{ anyways}} \cup \left(\frac{7}{9}, \frac{8}{9}\right)\right]$

$\to C_n = C_{n-1} - \bigcup_{k=0}^{n} \left(\frac{1+3k}{3^{n-1}}, \frac{2+3k}{3^{n-1}}\right)$

Thus: $C_n$ is closed (by induction)

$C = \bigcap_{n=1}^{\infty} C_n$ is closed. ∎

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Panel 5

⑥ $C$ is perfect.

$C$ is cpct + every point is an accumulation point.

Write $C_2 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$

$C_3 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$

$C_4 = [0, \frac{1}{27}] \cup [\frac{2}{27}, \frac{3}{27}] \cup \ldots$

Note: all endpoints are in $C$, length of each subinterval is $\frac{1}{3^n} \to 0$

Take $x \in C \Rightarrow x \in C_n \ \forall n \Rightarrow x$ is in one of

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Panel 6

of the subsets that makes up $C_n$. Take $x_n$ to be the left endpoint of that subset. If $x$ is already the left endpoint, take the right end point.

$\Rightarrow |x - x_n| < \frac{1}{3^n}$

Thus $x_n \to x$, $x_n \in C$

Thus: $x$ is accumulation point!

$\Rightarrow C$ is perfect.

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Panel 7

The Cantor set is uncountable but has length zero!

Proof: $C$ is perfect $\Rightarrow$ uncountable

$C_2 = C_1 - (\frac{1}{3}, \frac{2}{3})$, length $= 1 - \frac{1}{3}$

$C_3 = C_2 - \left[ (\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9}) \right]$, length $= 1 - \frac{1}{3} - \frac{2}{9}$

$C_4$, length $= 1 - \frac{1}{3} - \frac{2}{9} - \frac{4}{27}$

$= 1 - \left( \frac{1}{3} + \frac{2}{9} + \frac{4}{27} \right)$

length of $C_n = 1 - \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n}$ $\Rightarrow$ length of $C = 1 - \sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}}$

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Panel 8

$1 - \frac{1}{3} \sum_{n=0}^{\infty} \frac{2^n}{3^n} = 1 - \frac{1}{3} \frac{1}{1 - 2/3} = 1 - 1 = 0!$

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Panel 9

C contains no open set

Proof: Take any open set $U \subset C \Rightarrow \exists (a,b) \subset U \subset C$

$\Rightarrow (a,b) \subset C_n \ \forall n$

$\Rightarrow (a,b)$ is subset of one of subintervals of $C_n$
But length of the subintervals is $\frac{1}{3^n} \to 0$ so it can't contain $(a,b)$.

Also: if $(a,b) \subset C$ then $\text{length}(C) \geq b-a$
$\Rightarrow 0 \geq b-a \Rightarrow b=a$

↯

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Panel 10

We have seen that all endpoints are in C
But that's the least of the points in C.

$\text{card}(C) = c$, $\text{card}(\text{endpoints}) = \aleph_0$

$\Rightarrow \text{card}(C \text{ except endpoints}) = c$

Almost all $x \in C$ are like $\frac{1}{4}$, $0.\overline{3}$

Also: next to every point $p \in C$ is a $q \notin C$
because otherwise it would contain open interval.

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Panel 11

The Cantor Set is strange:

- uncountable, yet has length zero
- every point is accumulation point, yet does not contain any open set
- every point in $C$ is also an accumulation point of $C^c$

2D analog: Sierpinski carpet has zero area but infinite length

3D analog: Menger sponge has zero volume but inf. surface area

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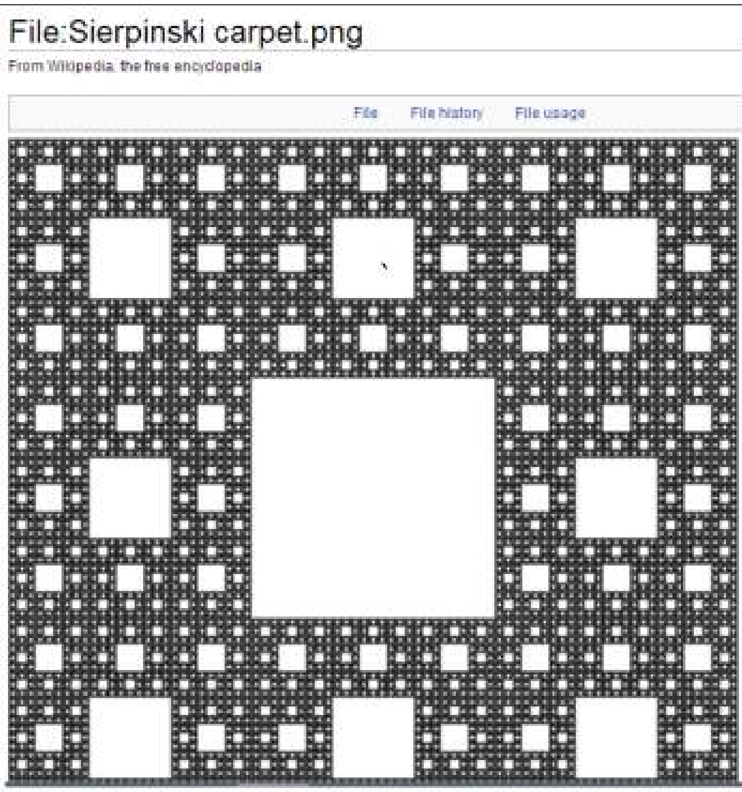
Panel 12

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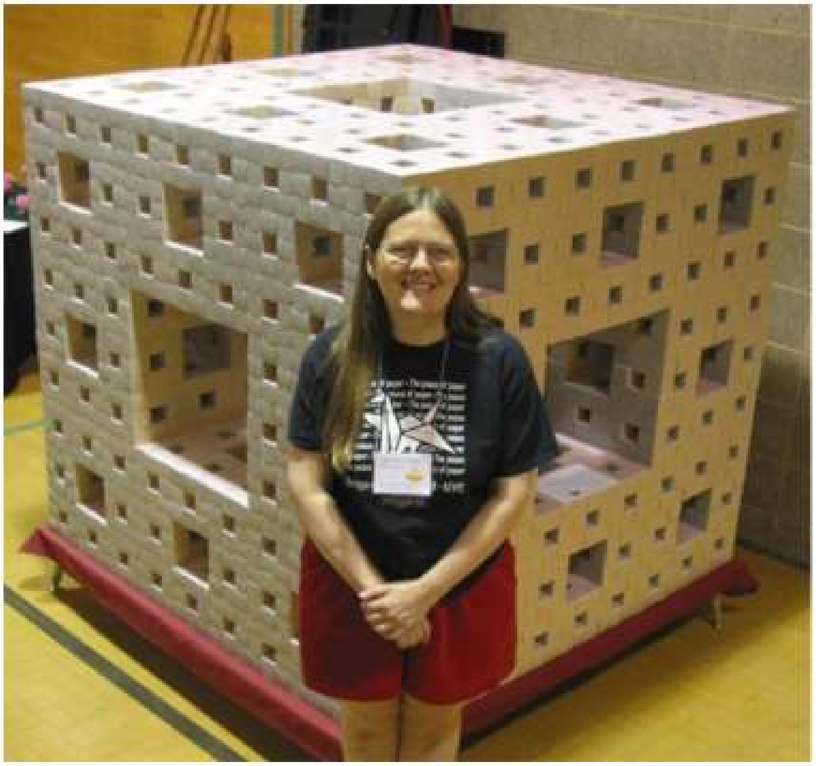
Panel 13



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Panel 14

Connected + Disconnected Sets

Def: An open set $S$ is disconnected if
$\exists\, U, V$ open s.t. $U \cap V = \emptyset$ and $U \cup V = S$

An arbitrary set $S$ is disconnected if
$\exists\, U, V$ open s.t. $(U \cap S) \cap (V \cap S) = \emptyset$
$(U \cap S) \cup (V \cap S) = S$

If $S$ is not disconnected it is connected

Ex: $\{ |x| < 1 ,\ x \neq 0 \} = (-1, 0) \cup (0, 1) = U \cup V$

$\{ |x| \leq 1 ,\ x \neq 0 \} = [-1, 0) \cup (0, 1]$ $\quad U = (-2, 0)$, $V = (0, 2)$

Both are disconnected!

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Panel 15

Is $\mathbb{Q}$ connected or disconnected?

Disconnected because $\mathbb{Q} = (U \cap \mathbb{Q}) \cup (V \cap \mathbb{Q})$
for $U = (-\infty, \pi)$, $V = (\pi, \infty)$
or $(-\infty, \sqrt{2})$, $V = (\sqrt{2}, \infty)$

Is $C$ (the Cantor set) connected or disconnected?

?

HW

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Panel 16

<u>Def:</u> A set $S$ is totally disconnected if for every $x, y \in S$ there are open sets $U, V$ s.t.

$$x \in U, \ y \in V$$

$$(U \cap S) \cap (V \cap S) = \emptyset$$

$$(U \cap S) \cup (V \cap S) = S$$

$(-1, 0) \cup (0, 1)$ disc. but not totally disc.

$\mathbb{Q}$ is totally disconnected.

$C$ is totally disconnected? $\leftarrow$ HW

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Panel 17

1) Find a totally disconnected set $S \subset [0,1]$ of length zero (different from Cantor set and $\mathbb{Q}$)

2) Find a totally disconnected set $S \subset [0,1]$ of length 1.

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