Panel 1

Def: A set $S \subset \mathbb{R}$ is compact: every sequence has a conv. subsequence whose limit is in A.

Heine - Borel Theorem:

in $\mathbb{R}^n$: compact = closed + bdd

Proof: Take S closed + bdd. take any sequence $\{a_n\} \subset S$.
$\{a_n\}$ is bdd $\Rightarrow$ by Bolzano-Weierstraß $\Rightarrow \exists\, a_{i_k} \to c$,
c is accumulation pt, S closed $\Rightarrow c \in S$
Take S compact: suppose not bdd. $\Rightarrow$ For every N
$\exists\, a$ s.t. $|a| > N$. Call that $\{a_N\} \to \infty$ $\Rightarrow$ every subseq

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Panel 2

converges to ∞. $\Rightarrow$ Not compact! Because $\infty \notin S$.
Suppose not closed. $\Rightarrow$ one accum. point outside S.
$\Rightarrow \exists\, a_j$ with $a_j \to c$, $c \notin S$
$\Rightarrow$ Every subsequence $\to c \notin S$ ↯

q.e.d

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Panel 3

Q: Find sets A_n s.t. (1) A_n are closed

(2) A_n are nested, i.e. $A_{n+1} \subset A_n$

(3) $\bigcap_{n=0}^{\infty} A_n = \emptyset$

$[0,n]$ $\quad$ $\underset{A_2}{[0,2]} \subset \underset{A_1}{[0,1]}$ NOT nested

$\Rightarrow [0,\frac{1}{n}]$ $\quad$ $[0,1] \supset [0,1/2] \supset [0,1/3]$ but $\bigcap [0,\frac{1}{n}] = \{0\} \neq \emptyset$

$\cancel{[\frac{1}{n^2},\frac{1}{n}]}$ $\quad$ $[1,1] \supset [\frac{1}{4},\frac{1}{2}]$ $\cancel{[\frac{1}{9},\frac{1}{3}]}$

$[n,\infty)$ $\quad$ $[1,\infty) \supset [2,\infty) \supset [3,\infty) \ldots \bigcap_{n=1}^{\infty} [n,\infty) = \emptyset$

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Panel 4

Thm: $A_n \subset \mathbb{R}$, A_n compact and nested, $A_{n+1} \subset A_n$ then

Then $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$

Proof: A_j compact $\Rightarrow \bigcap A_j$ is closed + bdd $\Rightarrow \bigcap A_j$ is cpct

Pick x_j in A_j. Know $\{x_j\} \subset A_1$. A_1 is compact,

$\exists\, x_{j_k} \to c$, $c \in A_1$

But $\{x_{j_k}\}_{k=2}^{\infty} \subset A_2$, has conv subsequence to $c \in A_2$

Next $\{x_{j_k}\}_{k=3} \subset A_3$, has conv subseqn to $c \in A_3$

$\Rightarrow c \in A_j\ \forall j \Rightarrow c \in \bigcap_{j=1}^{\infty} A_j \neq \emptyset$ qed

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Panel 5

Each A_j is compact, hence closed and bounded. Therefore, **A** is closed and bounded as well, and hence **A** is compact. Pick an $a_j \in A_j$ for each j.

Then the sequence $\{ a_j \}$ is contained in A_1. Since that set is compact, there exists a convergent subsequence $\{ a_{j_k} \}$ with limit in A_1.

But that subsequence, except the first number, is also contained in A_2. Since A_2 is compact, the limit must be contained in A_2.

Continuing in this fashion, we see that the limit must be contained in every A_j, and hence it is also contained in their intersection **A**. But then **A** can not be empty.

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Panel 6

Def: A set $S \subset \mathbb{R}$ is perfect if it is closed and every point is an accumulation point.

Ex: Find a perfect set $[0,5]$ ~~$\mathbb{N}$~~ $[0,1] \cup \{2\}$

Find a closed set that is not perfect

Find a compact set that is not perfect $\{1\}$

Find an unbdd closed set that is not perfect

$[0,\infty) \cup \{-1\}$

Find a closed set that is neither compact nor perfect.

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Panel 7

Thm. Every non-empty, perfect set is uncountable!

Ex: $[0,1]$ is uncountable

Proof: $[0,1]$ is perfect!

Proof: S is perfect. Assume $S = \{x_1, x_2, x_3, \ldots\}$ was countable. Find some $x \in S$ but not $x = x_i \; \forall i$

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