Panel 1

$$\sum_{n=1}^{\infty} (-1)^n \frac{n!}{n^n} \Rightarrow \left| (-1)^{n+1} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{1}{(-1)^n} \cdot \frac{n^n}{n!} \right|$$

$$= \cancel{(-1)} \left(\frac{(n+1)\, n^n}{(n+1)^{n+1}} \right)$$

$$\lim_{n\to\infty} \cancel{(-1)} \left(\frac{(n+1)\, n^n}{(n+1)^{n+1}} \right) \Rightarrow$$

$$\lim_{n\to\infty} \frac{\cancel{(n+1)}\, n^n}{\cancel{(n+1)}\, (n+1)^n} = \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^n =$$

$$= \lim \left(\frac{1}{\frac{1}{\frac{n}{n+1}}} \right)^n = \lim \left(\frac{1}{\frac{n+1}{n}} \right)^n =$$

$$= \lim_{n\to\infty} \frac{1}{(1+1/n)^n} = 1/e$$

$$\left(\frac{n^n}{2^n} \right) \to 0 \qquad \left(\frac{2^n}{n!} \right) \to 0 \qquad \left(\frac{n!}{n^n} \right)$$

converges!

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Panel 2

Let $\sum a_n = \sum (-1)^n \cdot \frac{1}{n^{1/2}}$ $\sum b_n = \sum (-1)^n \cdot \frac{1}{n^{1/3}}$ converge conditionally

$$\sum a_n b_n = \sum (-1)^n \frac{1}{n^{1/2}} \cdot (-1)^n \frac{1}{n^{1/3}} = \sum (-1)^{2n} \cdot \frac{1}{n^{5/6}} = \sum \frac{1}{n^{5/6}}$$

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Panel 3

Last Time: Topology

Open sets: $x \in S \Rightarrow U_\varepsilon$ of x, $U_\varepsilon \subset S$ where $U_\varepsilon = (x-\varepsilon, x+\varepsilon)$

Closed sets: S^c is open.

x boundary point of S: every nbhd contains points inside + outside of S

x is interior point of S: $\exists\, U_\varepsilon$ nbhd of x, $U_\varepsilon \subset S$

x is isolated point of S: $\exists\, U_\varepsilon$ nbhd of x s.t. $U_\varepsilon \cap S = \{x\}$

x is accumulation point of S: every U_ε nbhd of x contains inf. many points of S

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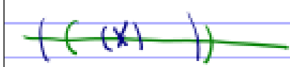
Panel 4

Ex: $(0,4)$ bdry : $[0,4]$

int. : $(0,4)$ all points



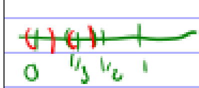
isolated : $\emptyset$

accum. : all of $[0,4]$

$\{1, 1/2, 1/3, 1/4, \ldots\} \cup \{\cancel{0}\}$:

bdry : itself $\cup\ \{0\}$

int. : $\emptyset$ every U_ε of $1/n$ contains irrationals



isol : $\{\frac{1}{n}\}$

accum. : $\{0\}$

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Panel 5

Thm. Every open set $S \subset \mathbb{R}$ is the union of countably many disjoint open intervals.

Proof: Two points $a, b \in S$ are related: if line segment $\overline{ab} \subset S$.

It is equiv. relation on S

$\Rightarrow S = \bigcup S_n$, S_n are disjoint

S_n is interval: take $a, b \in S_n \Rightarrow a \sim b$. take c between $a, b \Rightarrow c \sim a \Rightarrow c \in S_n \Rightarrow S_n$ is interval

S_n is open: Take $x \in S_n \Rightarrow x \in S \Rightarrow (x-\varepsilon, x+\varepsilon) \subset S$

$\Rightarrow$ every point in $(x-\varepsilon, x+\varepsilon) \sim x \Rightarrow$ in $S_n \Rightarrow S_n$ open

Countably many S_n only: HW

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Panel 6

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Panel 7

About bdry, isolated, etc. points:

a) Every point in S is either interior or bdry point.

b) For every S we have: $\text{bdry}(S) = \text{bdry}(S^c)$

c) A closed set contains all its bdry points
An open set contains no bdry points

d) Every non-isolated bdry point is an accumulation point.

e) An accumulation point is never isolated

f) S closed, c accum point $\Rightarrow c \in S$

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Panel 8

Circling back to Sequences

a) $S \subset \mathbb{R}$ is closed iff every Cauchy sequence has a limit that is inside S ($\Leftarrow$ HW)

b) Every bdd, infinite subset of $\mathbb{R}$ has accumulation point

c) If S is closed and bdd, and $\{a_n\} \subset S$, then $\{a_n\}$ has convergent subsequence with limit in S

Q: Find closed set S and $\{a_n\}$ in s.t. no subsequence converges to a point in S $\Rightarrow \{n\}$

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Panel 9

Thm: c is an accumulation point of S
$\Leftrightarrow \exists \{a_n\} \subset S$ with limit c.

Proof: 1) c is accum. point of S
a_1 is in S and in $(c-1, c+1)$
a_2 is in S and in $(c-\frac{1}{2}, c+\frac{1}{2})$
$\vdots$
clearly $\{a_n\} \to c$.

2) Suppose $\{a_n\} \subset S$, $a_n \to c \Rightarrow |a_n - c| < \varepsilon \ \forall n > N$
Thus, take that ε-nbhd. of c. $(c-\varepsilon, c+\varepsilon)$.
It contains $a_n \ \forall n \geq N \Rightarrow c$ is accumulation pt.

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Panel 10

Def: A set $S \subset \mathbb{R}$ is compact if every sequence $\{a_n\} \subset S$ has convergent subsequence, whose limit is in S.

Ex: $[0,1]$ compact

$[0,1)$ not compact: $\{1-\frac{1}{n}\} \in S$

$\{1,2,3\}$ compact

$\{n\}$ not compact

$\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ not cpct

$\{1, \frac{1}{2}, \frac{1}{3}, \ldots\} \cup \{0\}$ cpct

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Panel 11

Heine-Borel Theorem: $S \subset \mathbb{R}$ compact iff

S is closed + bdd.

Proof: next time

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