Panel 1

Last Time

Divergence Test: if $\{a_n\} \to 0$ NOT then $\sum a_n$ div.

Limit Comp. Test: $0 < \lim \left|\frac{a_n}{b_n}\right| < \infty$ then $\sum a_n$, $\sum b_n$ are the same

Geometric Series Test: $\sum r^n = \begin{cases} 1/_{1-r} & , |r|<1 \\ \text{undef. else} \end{cases}$

p-Series Test: $\sum 1/_{n^p}$ $\begin{cases} p \le 1 : \text{div.} \\ p > 1 : \text{conv.} \end{cases}$

Ratio Test: $\lim \frac{|a_{n+1}|}{|a_n|}$ $\begin{cases} <1 : \text{conv} \\ >1 : \text{div.} \\ =1 : \text{no info} \end{cases}$

Panel 2

Proof (the ratio test) Say $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$

$\exists \varepsilon$ s.t. $\left|\frac{a_{n+1}}{a_n}\right| < 1-\varepsilon \quad \forall n \geq N$

$\Rightarrow \quad |a_{N+1}| < (1-\varepsilon)\,|a_N|$

$|a_{N+2}| < (1-\varepsilon)|a_{N+1}| < (1-\varepsilon)^2 |a_N|$

$|a_{N+3}| < (1-\varepsilon)|a_{N+2}| < (1-\varepsilon)^3 |a_N|$ $(1-\varepsilon)<1$, so Geom. series conv.

$\vdots$

$|a_{N+j}| < (1-\varepsilon)^j |a_N| \Rightarrow \sum_{j=0}^{\infty} |a_{N+j}|$ conv. by comp. test with $\sum_{j=0}^{\infty} (1-\varepsilon)^j |a_N|$ #

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Panel 3

Root Test.

Consider $\sum_{n=0}^{\infty} a_n$

$\lim \sqrt[n]{|a_n|}$: $\begin{cases} <1 & \text{conv} \\ >1 & \text{div} \\ =1 & \text{no info} \end{cases}$

Root test is slightly better than Ratio test

Ex: $\sum_{n=1}^{\infty} 1/n^2$ $\lim_{n\to\infty} \sqrt[n]{n^{-2}} = \lim_{n\to\infty} (\sqrt[n]{n})^{2} = 1$ no info

But p-series with $p=2$ conv.

$\sum \frac{2^n}{n^2}$ $\lim_{n\to\infty} \sqrt[n]{2^n/n^2} = \lim_{n\to\infty} 2/\sqrt[n]{n^2} = 2$

diverges!

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Panel 4

Abel's Convergence Test

Consider the series $\sum a_n b_n$. Suppose

(1) $S_N = \sum_{n=1}^{N} a_n$ is bdd as a sequence

(2) $\{b_n\}$ is decreasing

(3) $\lim_{n\to\infty} b_n = 0$

Then series converges.

Proof is technical, and Test has one main application:

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Panel 5

Alternating Series Test

Consider $\sum_{n=0}^{\infty} (-1)^n b_n$ and $b_n \geq 0 \ \forall n$. Also:

$\{b_n\}$ is decreasing, and $\lim_{n\to\infty} b_n = 0$. Then it converges

Ex: $\sum_{n=0}^{\infty} (-1)^n / (n+1)$ $\quad$ $b_n = 1/(n+1)$ it decreasing to zero.

Proof: $a_n = (-1)^n$. Then $S_N = \sum_{n=0}^{N} (-1)^n$: $1,\ 1-1,\ 1-1+1,\ 1-1+1-1$

$1,\ 0,\ 1,\ 0$

is bdd

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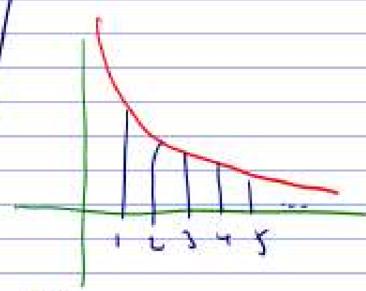
Panel 6

Important Tests

- Divergence Test (*)
- Comparison Test
- Limit Comparison Test (*)
- Cauchy Condensation Test
- Geometric Series Test (*)
- p Series Test (*)
- Root Test
- Ratio Test (*)
- Abel's Convergence Test
- Alternating Series Test (*)
- Integral Test

f positive + decreasing



$$\int_1^{\infty} f(x)\,dx \simeq \sum_{j=1}^{\infty} f(x_j)\, 1$$

$$= \sum_{j=1}^{\infty} a_j$$

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Panel 7

$\sum_{n=1}^{\infty} \frac{1}{n^{3/4}}$ p-series, $p=3/4$, div

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ p series, $p=2$. conv.

$\sum_{n=1}^{\infty} \frac{n}{1+n}$ div. by Div test

$\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n^2}$ div Comp. test. $|a_n| \leq |b_n|$

$\frac{n-1}{n^2} > \frac{1}{n}$ d.n.e.

$\sum_{n=1}^{\infty} \frac{n+1}{n(n+2)}$ $\lim \frac{n-1}{n^2} \cdot \frac{n}{1} = 1$ limit comp test applies $\Rightarrow$ div

$\sum_{n=1}^{\infty} \frac{4}{3^n}$ Geom. $r = (\frac{1}{3})$, conv.

$\sum_{n=1}^{\infty} \frac{4^n}{3^n}$ Geom. $r = (\frac{4}{3})$ div.

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Panel 8

$\sum_{n=1}^{\infty} \frac{1}{3^n - 5}$

$\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2 - 3}$ Not conv. abs. $\left(\sum \left|\frac{(-1)^n n}{n^2-3}\right| = \sum \frac{n}{n^2-3} \sim \sum \frac{1}{n}\right)$

$\sum_{n=1}^{\infty} \frac{(-1)^n (n+1)}{\sqrt{n}}$ $\frac{n}{n^2-3} \to 0$, decr. $\Rightarrow$ conv.

is conditionally convergent!

$\sum_{n=1}^{\infty} \frac{n}{2^n}$ $\frac{n+1}{2^{n+1}} \Big/ \frac{n}{2^n} = \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{1}{2} \frac{n+1}{n} \to \frac{1}{2}$

$\sum_{n=1}^{\infty} \frac{n!}{3^n}$

$\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ $\frac{2^n}{n^3} \cdot \frac{1}{2^n} \to 0$

$\sum_{n=1}^{\infty} \frac{n+1}{2n-1}$

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Panel 9

⊗ $\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+2}$

$\sum_{n=1}^{\infty} \frac{1}{n(n+3)}$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[n]{n}}$

$\sum_{n=1}^{\infty} \frac{ln(n)}{n}$

$\sum_{n=1}^{\infty} \frac{(2n)!}{n^5}$

$\sum_{n=1}^{\infty} \frac{(2n)!}{(2n+1)!}$

$\frac{n+2-n}{n(n+2)} = \frac{2}{n(n+2)}$ conv. by limit comp with $\sum 1/n^2$

$\frac{2}{n(n+2)} \cdot \frac{n^2}{1} \to 2$ so lim

Both conv.!

What is the limit of ⊗

$\sum \frac{1}{n} - \frac{1}{n+2} = 1 - \cancel{\frac{1}{3}} + \frac{1}{2} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}$

Telescoping Series

$S_N = 1 - \cancel{\frac{1}{3}} + \frac{1}{2} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{6}} \dots + \cancel{\frac{1}{N}} - \frac{1}{N+2}$

$= 1 + \frac{1}{2} - \frac{1}{N+2}$

A) conv. abs

B) cond. conv.

div

Panel 10

$\frac{\ln(n)}{n} \Big/ \frac{1}{n} = \frac{\ln(n)}{\cancel{n}} \cdot \cancel{n} \to \infty$

$\frac{\ln(n)}{n} \geq \frac{1}{n}$

Integral test: $\int_2^{\infty} \frac{\ln(x)}{x}\, dx$ conv./diverg