Panel 1

$$\sum_{n=12}^{\infty}\left(\frac{9}{11}\right)^n = \left(\frac{9}{11}\right)^{12} + \left(\frac{9}{11}\right)^{13} + \left(\frac{9}{11}\right)^{14} + \dots$$

$$= \left(\frac{9}{11}\right)^{12}\left(\underbrace{1 + \frac{9}{11} + \left(\frac{9}{11}\right)^2 + \dots}_{\text{know } \left(\frac{1}{1-9/11}\right)}\right)$$

$$= \left(\frac{9}{11}\right)^{12} \cdot \frac{11}{2}$$

$$\sum r^n = \frac{1}{1-r}, \ |r|<1$$

12 done

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \dots$$

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} - \frac{1}{4}$$

1

Panel 2

Series: Convergence Tests

Cauchy Criterion for Series:

$\sum_{j=1}^{\infty} a_j$ converges iff $\forall \varepsilon > 0$ there is N s.t. whenever $n > m \geq N$ then

$$\left|\sum_{j=n}^{m} a_j\right| < \varepsilon$$

Proof: If $|S_m - S_n| = \left|\sum_{j=n}^{m} a_j\right| < \varepsilon$, $\{S_n\}$ is Cauchy, hence converges

#

2

Panel 3

Divergence Test: If $\sum a_n$ converges, $\lim_{n\to\infty} a_n = 0$

Ex: $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$ ~~not~~ applies, if $\sum$ did conv.
$\lim \left(1+\frac{1}{n}\right)^n = 0$ which it is not!

Proof:

$S_n - S_{n-1} = a_n$

$\lim S_n - S_{n-1} = \lim a_n$ $\qquad$ $\lim S_n = L$ by assumpt.

$0 = L - L = \lim_{n\to\infty} a_n$

#

3

Panel 4

Comparison Test:

(1) Suppose $\sum b_n$ converges absolutely, and
$|a_n| \le |b_n|$ Then $\sum a_n$ conv. absolutely

(2) Suppose $\sum a_n$ diverges to $+\infty$, and
$b_n \ge a_n$ then $\sum b_n$ diverge as well.

Trick: $|a_n| \le |b_n|$ — if $\sum b_n$ conv. $\Rightarrow \sum a_n$ conv
if $\sum a_n$ div $\Rightarrow \sum b_n$ div.

Ex: $\sum_{n=2}^{\infty} \frac{1}{n-1}$ $\qquad$ $\frac{1}{n-1} > \frac{1}{n}$ $\qquad$ $\sum \frac{1}{n} = \infty$ thus
$\sum \frac{1}{n-1} = \infty$

4

Panel 5

$\sum \frac{1}{n+1}$, $\frac{1}{n+1} \leq \frac{1}{n}$

$\Rightarrow \sum \frac{1}{n+1} \leq \sum \frac{1}{n} = \infty$

Comp. test does apply to $\sum \frac{1}{n+1}$, but not to $\sum 1/{n+1}$ (which also diverges)

5

Panel 6

Limit Comparison Test

Suppose $\{a_n\}$, $\{b_n\}$ are two sequences. Define $r = \lim \left|\frac{a_n}{b_n}\right|$. If $0 < r < \infty$ then $\sum a_n$ conv. abs iff $\sum b_n$ conv. abs

Ex: $\sum_{n=1}^{\infty} \underbrace{\frac{n+5}{n^2-7n+1}}_{a_n} \sim \sum \underbrace{\frac{1}{n}}_{b_n}$ Thus, Sol. diverge

$\lim_{n\to\infty} \frac{n+5}{n^2-7n+1} \cdot \frac{n}{1} = 1$ $\sum \frac{1}{n+1} \sim \sum \frac{1}{n}$ $\frac{n}{n+1} \to 1$

6

Panel 7

Proof. $r = \lim \frac{|a_n|}{|b_n|}$

$\Rightarrow \exists\ c, C$ st. $c \le \left|\frac{a_n}{b_n}\right| \le C \quad \forall n \ge N$

Thus $c|b_n| \le |a_n|$ by comp. if $\sum a_n$ conv, $\sum b_n$ conv

$|a_n| \le C|b_n|$ if $\sum a_n$ div., $\sum b_n$ div.

#

7

Panel 8

Geometric Series Test.

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \dots \begin{cases} \text{conv. if } |r|<1 \\ \text{div. if } |r| \ge 1 \end{cases}$$

Moreover $\sum_{n=0}^{\infty} r^n = 1/(1-r)$, $|r|<1$. (know the actual limit! Rare)

Proof. $S_N = 1 + r + r^2 + \dots + r^N$

$r S_N = \cancel{r} + \cancel{r^2} + \dots + r^N + r^{N+1}$

$S_N - r S_N = 1 - r^{N+1}$

$S_N(1-r) = 1 - r^{N+1} \quad \Rightarrow \quad S_N = \frac{1 - r^{N+1}}{1-r}$

$r^{N+1} \to 0$ if $|r|<1$

$\to \infty$ if $|r|>1$

$r=-1$: $\sum_{n=0}^{\infty} (-1)^n$ div. by div. test. also $\sum r^n$, $|r| \ge 1$

8

Panel 9

Geometric Series is one of the very few series where we actually know the limit:

Ex: $\sum_{n=0}^{\infty} \left(\frac{7}{8}\right)^n$ ✓

$\sum_{n=10}^{\infty} \left(\frac{3}{4}\right)^n$ ✓ $= \left(\frac{3}{4}\right)^{10} \frac{1}{1-3/4}$

$\sum_{n=1}^{\infty} \frac{2^n+3}{3^n-7}$ limit-compare with $\sum \frac{2^n}{3^n} = \sum_0^{\infty} \left(\frac{2}{3}\right)^n$, conv $= \frac{1}{1-2/3}$

$\frac{2^n+3}{3^n-7} \cdot \frac{3^n}{2^n} \to 1$ limit comp. applies!

9

Panel 10

p - Series Test: $\sum_{n=1}^{\infty} 1/n^p$ is called a p-series

if $p \le 1$: div

if $p > 1$: conv.

Ex: $\sum \frac{1}{1+n^2}$ conv. by limit comp with p-series, $p=2$

$\sum \frac{1}{3-\sqrt{n}}$ div. by limit comp with p series, $p=1/2$

$\sum \frac{1}{n^{1.0001}}$

10

Panel 11

Ratio Test: Consider $\sum_{n=0}^{\infty} a_n$. works well for factorials

$\lim \left|\frac{a_{n+1}}{a_n}\right| < 1 \Rightarrow$ conv.

$\lim \left|\frac{a_{n+1}}{a_n}\right| > 1 \Rightarrow$ div

$\lim \left|\frac{a_{n+1}}{a_n}\right| = 1$: wasted time

Note: The proper conditions involve lim sup!

Ex: $\sum_{n=0}^{\infty} 1/n!$ conv. $\frac{a_{n+1}}{a_n} = \frac{1/(n+1)!}{1/n!} = \frac{n!}{(n+1)!} = \frac{n!}{(n+1)n!} = \frac{1}{n+1} \to 0$

$\sum_{n=0}^{\infty} \frac{n^2}{2^n}$ conv $\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} = \left(\frac{n+1}{n}\right)^2 \cdot \frac{1}{2} \to 1/2$

11

Panel 12

$\sum_{n=0}^{\infty} \frac{2^n}{n!}$ $\qquad \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \frac{2}{n+1} \to 0$

So $\sum \frac{2^n}{n!}$ converges

$\Rightarrow \lim \frac{2^n}{n!} = 0$ which means fact. grows even faster that exp.

12

Panel 13

Proof: next line

13

Panel 14

Root Test. Consider $\sum_{n=0}^{\infty} a_n$.

$\lim_{n\to\infty} |a_n|^{1/n} < 1$: conv

$\lim_{n\to\infty} |a_n|^{1/n} > 1$: div

$\lim |a_n|^{1/n} = 1$: no info!

Ex: $\sum_{n=1}^{\infty} 1/n^2$ HW

$\sum \frac{2^n}{n^2}$

14

Panel 15

Abel's Convergence Test

Consider the series $\sum a_n b_n$. Suppose

next time

Then series converges.

15