Panel 1

Series: The formal expression $\sum_{j=1}^{\infty} a_j = a_1 + a_2 + a_3 + \dots$ is called series. The expression

$$S_N = \sum_{j=1}^{N} a_j = a_1 + a_2 + \dots + a_N$$

is called N-th partial sum. If $\lim_{N \to \infty} S_N$ exists, then the series is called convergent.

Note that $\{S_N\}$ is the sequence of partial sums

Ex: $\sum_{j=1}^{\infty} 1$. $S_1 = 1$, $S_2 = 1+1 = 2$, $S_3 = 3, \dots$, $S_N = N$

$\sum 1$ diverges

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Panel 2

$$\sum_{j=1}^{\infty} \left(\frac{1}{2}\right)^j = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$

$S_1 = 1/2$, $S_2 = 1/2 + 1/4$, $S_3 = 1/2 + 1/4 + 1/8, \dots$

$$S_N = \frac{1}{2} + \cancel{\frac{1}{4}} + \cancel{\frac{1}{8}} + \cancel{\dots} + \cancel{\frac{1}{2^{N-1}}} + \cancel{\frac{1}{2^N}}$$

$$\frac{1}{2} S_N = \cancel{\frac{1}{4}} + \cancel{\frac{1}{8}} + \cancel{\frac{1}{16}} + \cancel{\dots} + \cancel{\frac{1}{2^N}} + \frac{1}{2^{N+1}}$$

$$S_N - \frac{1}{2} S_N = \frac{1}{2} - \frac{1}{2^{N+1}}$$

$$\frac{1}{2} S_N = \left(\frac{1}{2} - \frac{1}{2^{N+1}}\right) \Rightarrow S_N = 2\left(\frac{1}{2} - \frac{1}{2^{N+1}}\right) = 1 - \frac{1}{2^N}$$

$$\lim S_N = 1$$

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Panel 3

$\sum 1$ diverges $\quad \sum 1 = 1+1+1+\ldots$

$\sum \left(\frac{1}{2}\right)^n$ converges! $\quad \sum \left(\frac{1}{2}\right)^n = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots$

Conjecture: if term to add get smaller series converges

$\sum_{n=1}^{\infty} \frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots$ diverges!!!! Too Sad!

$S_1 = 1 = 1$

$S_2 = 1+1/2 = 1+1/2$

$S_4 = 1+1/2+(1/3+1/4) \geq 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right) = 1+2\,1/2$

$S_8 = 1+1/2+1/3+1/4+1/5+1/6+1/7+1/8 \geq 1+\frac{1}{2}+\underbrace{\frac{1}{4}+\frac{1}{4}}+\underbrace{\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}} = 1+3/2$

$S_{2^n} \geq 1+n/2$ Thus, subsequence S_{2^n} of S_n diverges!

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Panel 4

Thus, sequence can not converge $\#$

Note $\sum_{n=1}^{\infty} \frac{1}{n}$ is called Harmonic Series. It diverges!!

Def: The series $\sum_{n=1}^{\infty} a_n$ converges absolutely, if $\sum_{n=1}^{\infty} |a_n|$ converges. If $\sum a_n$ converges, but not absolutely, $\sum a_n$ is called conditionally convergent.

Ex: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\ldots$

is called alternating harmonic series.

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Panel 5

$\sum_{n=0}^{\infty} (-1)^n$ is an alternating series

$= (1-1)+(1-1)+(1-1)+(1-1)+\dots = 0$

$= 1+(-1+1)+(-1+1)+(-1+1)+(-1+1)+\dots = 1$

$1+1+(-1+1)+(-1+1)+(-1 \dots = 2$

<u>Thm</u>: The alternating harmonic series is conditionally convergent, i.e. $\sum_{n=1}^{\infty} (-1)^{n+1}/n$ converges, but

$\sum \left|\frac{(-1)^{n+1}}{n}\right| = \sum 1/n$ diverges.

<u>Note</u>: Take any $x \in \mathbb{R}$. There is some rearrangement of alternating harmonic series that converges to x.

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Panel 6

<u>Ex</u>: Make $\sum \frac{(-1)^{n+1}}{n}$ converge to, say, 2.

Want: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} \pm \dots = 2$

$\underbrace{1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots + 1/15}_{2.021} \underbrace{-1/2}_{1.521} + 1/17 + 1/19 + 1/21 + \dots - 1/4 \rightsquigarrow 2$

Works because (a) $\frac{1}{2n+1} + \frac{1}{2n+3} + \frac{1}{2n+5} + \dots = \infty$ so that

I can always go above 2!

(b) I subtract $\frac{1}{2n} \to 0$

$\# + \infty = \infty$, $\infty + \infty$, $-\infty + \infty$ undef.!

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Panel 7

Thm: · If $\sum a_n$ is abs. convergent, then any rearrangement of a_n will converge to the same limit

· If $\sum a_n$ is cond. convergent and L is any number, then some rearrangement will converge to L.

"(Conditionally convergent) series are the work of the devil!"
Divergent
Sig N.H. Abel

Thm: If two series $\sum a_n$ and $\sum b_n$ conv. absolutely, then we can add, subtract, mult., div. them as expected.

Not so for cond. convergent series!

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Panel 8

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = 0$$ is only one arrangement!

Need Conv. Tests

Essential	Others
Divergence	Comparison test
Limit Comparison	Cauchy condensation
Geometric Series	Root test
p-Series	Abel's Convergence Test
Ratio test	Integral test
Alternating series test	

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Panel 9

Divergence Test. If $\sum_{n=0}^{\infty} a_n$ converges, then $\{a_n\} \to 0$

or: if $a_n \not\to 0$ then $\sum a_n$ diverges

Ex: $\sum_{n=1}^{\infty}\left(1-\frac{1}{n}\right)$. $\{1-\frac{1}{n}\} \to 1 \neq 0$, so $\sum\left(1-\frac{1}{n}\right)$ diverges

Limit Comp Test: If $\sum a_n$ and $\sum b_n$ are two series that are 'comparable', i.e. $\limsup \left|\frac{a_n}{b_n}\right| = r$, and $0 < r < \infty$, then either both series conv. abs. or both diverge.

Ex: $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$, $\sum_{n=1}^{\infty} \frac{1}{2^n}$ check $\lim \left|\frac{a_n}{b_n}\right| = \lim_{n\to\infty}\left|\frac{2^n}{2^n-1}\right| = 1$

a_n converges b_n by limit comp test

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Panel 10

Geometric Series test: $\sum_{n=0}^{\infty} r^n$: $\begin{cases} \text{converges abs to } \frac{1}{1-r} & \text{if } |r|<1 \\ \text{diverges} & \text{if } |r| \geq 1 \end{cases}$

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