Panel 1

Real Analysis - Last time

Cauchy Sequences: $|a_j - a_k| < \varepsilon \quad \forall j, k \geq N$.

Completeness Thm: Every Cauchy sequence in $\mathbb{R}$ must converge.

Subsequences: take some a_j, not others

subsequences and convergence

if $\{a_n\}$ conv. $\Rightarrow \{a_{j_n}\}$ converges to L

if all subsequences $\{a_{j_n}\}$ converge to L, then $\{a_n\} \to L$

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Panel 2

Big Deal Theorem: (Bolzano-Weierstraß Thm)

Every bdd sequence $\{a_j\}$ has a convergent subsequence!

Ex: $\{(-1)^n\}$ bdd $\Rightarrow -1, -1, \ldots \to -1$ and $1, 1, 1, \ldots \to 1$

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Panel 3

Proof of Bolzano-Weierstraß Theorem

$\{a_n\}$ is bdd $\Rightarrow |a_j| \le M \ \forall j$.

either $[-M, 0]$ or $\underline{[0, M]}$ contain infinitely many a_j

say $[0, M]$. Pick $a_{j_1} \in [0, M]$

either $[0, \frac{M}{2}]$ or $\underline{[\frac{M}{2}, M]}$ contain infinitely many a_j's

say $[\frac{M}{2}, M]$. pick $a_{j_2} \in [\frac{M}{2}, M]$

either $[\frac{M}{2}, \frac{3M}{4}]$ or $[\frac{3M}{4}, M]$ contain infinitely many a_j's

pick a_{j_3} in that interval

Keep on going $\Rightarrow$ Get $\{a_{j_k}\}$ a subsequence

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Panel 4

$|a_{j_1} - a_{j_2}| \le M$

$|a_{j_2} - a_{j_3}| \le M/2$

$|a_{j_3} - a_{j_4}| \le M/4$

$|a_{j_k} - a_{j_{k+1}}| \le M/2^{k-1}$ $\Rightarrow$ consecutive elements are closer

$$|a_{j_k} - a_{j_m}| \le |a_{j_k} - a_{j_{k+1}}| + |a_{j_{k+1}} - a_{j_{k+2}}| + |a_{j_{k+2}} - a_{j_{k+3}}| + \dots + |a_{j_{m-1}} - a_{j_m}|$$
$$\le M/2^{k-1} + M/2^k + M/2^{k+1} + \dots + M/2^{m-2}$$
$$= \frac{M}{2^{k-1}}\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^{m-k-1}}\right)$$

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Panel 5

Know: (*) $1+\frac{1}{2}+\frac{1}{4}+\dots\frac{1}{2^j} \le 1+\frac{1}{2}+\frac{1}{4}+\dots = 2$

Thus: $|a_{j_k} - a_{j_m}| \le \frac{M}{2^{k+1}} \cdot 2 = \frac{M}{2^{k-2}}$

Given any $\varepsilon > 0$, choose N s.t. $\frac{M}{2^{N-2}} < \varepsilon$

$\Rightarrow |a_{j_k} - a_{j_m}| < \frac{M}{2^{N-2}} < \varepsilon$ if $j_k, j_m \ge N$

Thus $\{a_{j_k}\}$ is Cauchy, and by completeness of $\mathbb{R}$ $\{a_{j_k}\}$ converges!

(*) see later slide.

qed

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Panel 6

Last weapon to get sequences to fall in line: lim sup and lim inf

Def: Let $\{a_j\}_{j=1}^{\infty}$ be a sequence of real numbers. Define

$A_j = \inf\{a_j, a_{j+1}, a_{j+2}, \dots\}$

and let $c = \lim(A_j)$. Then c is called the **limit inferior** of the sequence $\{a_j\}_{j=1}^{\infty}$.

Let $\{a_j\}_{j=1}^{\infty}$ be a sequence of real numbers. Define

$B_j = \sup\{a_j, a_{j+1}, a_{j+2}, \dots\}$

and let $c = \lim(B_j)$. Then c is called the **limit superior** of the sequence $\{a_j\}_{j=1}^{\infty}$.

In short, we have:

1. $\lim\inf(a_j) = \lim(A_j)$, where $A_j = \inf\{a_j, a_{j+1}, a_{j+2}, \dots\}$
2. $\lim\sup(a_j) = \lim(B_j)$, where $B_j = \sup\{a_j, a_{j+1}, a_{j+2}, \dots\}$

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Panel 7

Examples: Find lim sup and lim inf for

a) $\{(-1)^j\}$

$A_1 = \inf\{\cancel{-1}, 1, -1, 1\} = -1$ $A_j = -1$.

$A_2 = \inf\{1, -1, 1, -1\} = -1$ lim inf a_j =

lim sup $(-1)^j = 1$ lim $A_j = -1$

b) $\{1/j\}$

$\{1/j\} = \{1, 1/2, 1/3, 1/4, \ldots\}$

$A_1 = \inf(1, 1/2, 1/3, 1/4, \ldots) = 0$

$A_2 = \inf(1/2, 1/3, 1/4, \ldots) = 0$

$A_3 = \inf(1/3, 1/4, 1/5, \ldots) = 0$

$\Rightarrow$ lim inf $(a_j) = 0$

$B_1 = \sup(1, 1/2, 1/3, \ldots) = 1$

$B_2 = \sup(1/2, 1/3, 1/4, \ldots) = 1/2$

$B_3 = 1/3$

lim sup $(a_j) = \lim_{j\to\infty} B_j =$

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Panel 8

c) $\{(-1)^j \cdot j\} = \{-1, 2, -3, 4, -5, 6, \ldots\}$

lim inf $(a_j) = -\infty$

lim sup $(a_j) = \infty$

d) $\{(-1)^j / j\} = \{-1, 1/2, -1/3, 1/4, -1/5, 1/6, \ldots\}$

lim inf $(a_j) = 0$

lim sup $(a_j) = 0$

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Panel 9

Prop: (1) lim sup and lim inf always exist (possibly $\pm\infty$) for any sequence.

(2) If $\lim a_i = L$ then $\limsup(a_i) = \liminf(a_i) = L$

(1) Look at lim sup: A_j are increasing. SO THERE.

$B_1 = \sup(a_1, a_2, \ldots)$

$B_2 = \sup(a_2, a_3, \ldots)$

$B_3 = \sup(a_3, a_4, \ldots)$

$\Rightarrow B_j$ are decreasing. If bdd $\Rightarrow$ converges. Else unbdd $\Rightarrow \lim = -\infty$

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Panel 10

Special Sequences:

$\{a^n\}$ Power sequence. If $a > 1$: goes to ∞. If $|a| < 1$: goes to zero. $a = 1$: conv. to 1. $a \leq -1$: no limit

$\{n^\alpha\}$ Exponent sequence. $\alpha > 0 \Rightarrow$ goes to ∞, $\alpha < 0$, goes to zero, $\alpha = 0$. const

$\{n^{1/n}\} = \{1, \sqrt[2]{2}, \sqrt[3]{3}, \sqrt[4]{4}, \ldots\} \to 1$ (Root n of n seq)

$\{a^{1/n}\}$ n-th Root sequence $\to 1$ if $a > 0$ ✓

$\left\{\frac{n^k}{b^n}\right\}$ Binomial sequence $(b > 0)$, conv. to 0!

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Panel 11

Chapter 4: Series

Want to add: $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4} = \frac{1}{4}+\frac{1}{3}+\frac{1}{2}+1$

Add inf. many numbers: $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots$ gives ∞ ?

Zeno's Paradox: Achilles runs against turtle.

A. runs 10 m/sec, T runs 5 m/sec

A give T 10 meter head-start on 100 m track

Both run: After 1 sec. A covers 10 m, is at spot where T. started. T is 5 m. ahead

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Panel 12

A. runs to where T is, reaches in. Sut T is 2.5 m up.

A. covers 2.5 meters, T is now 1.25 m ahead.

$\Rightarrow$ A never passes Turtle

time	difference
$t=0$	10
1	5
$1+\frac{1}{2}$	2.5
$1+\frac{1}{2}+\frac{1}{4}$	1.25
$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$	0.625

A will reach T as $n\to\infty$, i.e. in

$$T = 1+1/2+1/4+1/8+\dots$$

$$T_n = 1+\cancel{1/2}+\cancel{1/4}+\cancel{1/8}+\dots+\cancel{1/2^n}$$

$$\frac{1}{2}T_n = \cancel{1/2}+\cancel{1/4}+\cancel{1/8}+\dots+1/2^{n+1}$$

$$T_n-\frac{1}{2}T_n = 1-1/2^{n+1} \Rightarrow \frac{1}{2}T_n = 1-1/2^{n+1}$$

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Panel 13

$T_n = 2\left(1 + 1/2^{n+1}\right)$ Thus $\lim T_n = 2$
because $1/2^{n+1}$ converges to zero!

Thus, after 2 secs difference is 0 and
A passes T just fine after 2 secs!

Def: The (formal) expression $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$
is called (infinite) series.
The expression $S_N = a_1 + a_2 + \ldots + a_N$ is called
n-th partial sum
If $\lim S_N$ exists, then the series converges,
otherwise diverges. Note: $\{S_N\}$ is a sequence.

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