Panel 1

Real Analysis - Last time

Monotone Sequences

Recursive Sequences

Bounded Sequences

Bdd + monotone ⇒ convergence

1

Panel 2

Thm: $(1+\frac{1}{n})^n$ converges.

Read this!

2

Panel 3

Problem with convergent sequences: need to know the limit L in advance!

Def: A sequence $\{a_n\}$ is called Cauchy if for every $\varepsilon > 0$ $\exists N (= N(\varepsilon))$ s.t.

$$|a_j - a_k| < \varepsilon \quad \text{if } j, k \geq N$$

Cauchy: if terms get closer to each other as j, k increase

3

Panel 4

Completeness Thm: Let $\{a_j\}$ be a sequence of real numbers. Then

$$\{a_n\} \text{ is Cauchy iff } \{a_n\} \text{ converges}$$

Note: Not true in $\mathbb{Q}$: $\{1, 1.4, 1.41, 1.414, \ldots\}$ $(\to \sqrt{2})$

Proof: Suppose $a_n \to L$. $\Rightarrow$ given $\varepsilon > 0$ $\exists N$ s.t.

$$|a_j - L| < \varepsilon/2 \quad \text{if } j \geq N$$

$$|a_j - a_k| = |a_j - L + L - a_k|$$
$$\leq |a_j - L| + |a_k - L| \leq \underline{\underline{\tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon}} \quad \text{if } j, k \geq N$$

q.e.d.

4

Panel 5

<u>Lemma:</u> Say $\{a_j\}$ is Cauchy. Then $\{a_j\}$ is bounded.

[Proof]: $|a_j - a_k| < 1 \quad \forall j,k \geq N$

$$|a_j| = |a_j - a_k + a_k| \leq |a_j - a_k| + |a_k|$$
$$\leq 1 + |a_k| \quad \forall j,k \geq N$$

$\Rightarrow |a_j| \leq 1 + |a_N|, \quad j \geq N$

Pick $M = \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, 1 + |a_N|\}$

$\Rightarrow |a_j| \leq M \quad \forall j$

5

Panel 6

Suppose $\{a_n\}$ is Cauchy. Then define

$$S = \{x \in \mathbb{R} : x < a_j \text{ for all but finitely many } j\}$$

$$x = \sup(S)$$

Ex: $\{a_n\} = \{-3, -2, -1, 0, 1/2, 1/3, 1/4, 1/5, 1/6, \ldots\}$

$\frac{1}{7} \notin S$ because $\frac{1}{7} < a_j$ true for $\frac{1}{4}, \frac{1}{3}, \frac{1}{2}$
not true for $1/6, 1/6 \ldots$

$-1 \in S$: $-1 < a_j$ true from $0, 1/2, 1/3, 1/6 \ldots$
not true for $-3, -2$

6

Panel 7

$S = \{x \in \mathbb{R} : x < a_j \text{ for all but finitely many } j\}$

Know $|a_i| \leq M \Rightarrow -M \leq a_j \leq M \quad \forall j$

$\Rightarrow -M \in S$, M upper bdd.

$\Rightarrow S \neq \emptyset$, bdd above $\Rightarrow$ lub-property.

$x = \sup(S) \Rightarrow x \leq M$

$-M \leq x \leq M$

7

Panel 8

$S = \{x \in \mathbb{R} : x < a_j \text{ for all but finitely many } j\}$

$x = \sup(S)$. Know $-M < x < M$ for some M.

Take $\varepsilon > 0$. $\exists N$ s.t. $|a_j - a_k| < \varepsilon/2 \quad \forall j, k > N$

$\Rightarrow |a_j - a_{N+1}| < \varepsilon/2 \Rightarrow -\varepsilon/2 < a_j - a_{N+1} < \varepsilon/2$

$\Rightarrow a_j > a_{N+1} - \varepsilon/2 \quad \forall j > N$

$\Rightarrow a_{N+1} - \varepsilon/2 \in S \Rightarrow a_{N+1} - \varepsilon/2 \leq x$

Also $a_j < a_{N+1} + \varepsilon/2 \quad \forall j > N \Rightarrow a_{N+1} + \varepsilon/2 > x$

$a_{N+1} + \varepsilon/2 \notin S$

$\Rightarrow |a_{N+1} - x| < \varepsilon/2$

8

Panel 9

Finally: $|a_j - x| \le |a_j - a_{N+1}| + |a_{N+1} - x|$

$< \varepsilon/2 + \varepsilon/2 = \varepsilon \quad \forall j \geq N$

$a_j \to x$

q.e.d

9

Panel 10

Another problem with sequences is that not all of them converge. Solution: extract subsequences

Def: Take $\{a_n\}$. Extract some terms, drop others $\Rightarrow \{a_{j_k}\}$ is a subsequence.

Ex: $\{(-1)^j\}$ extract 2 convergent subsequences

$\{-1, 1, -1, 1, -1, 1\}$ $\{-1\}$ and $\{1\}$

$\{1/n\}$ extract any 3 subsequences

$\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ $\{\frac{1}{2n+1}\} \to 0$ $\{\frac{1}{2n}\} \to 0$, $\{a_{j_k}\} \to 0$

10

Panel 11

Thm: If $\{a_n\}$ converges to a limit L then every subsequence $\{a_{j_k}\}$ converges to L also.

Conversely, if every subsequence $\{a_{j_k}\}$ of $\{a_j\}$ converges to the same limit L, then $a_j \to L$ also.

How do I show that $\{(-1)^n\}$ does not conv.?

$\{-1\} \to -1$, $\{1\} \to 1$ $\Rightarrow$ $\{(-1)^n\}$ can't conv.

Proof: Suppose $a_j \to L$ $\Rightarrow$ $\forall \varepsilon\ \exists N$ s.t.

$|a_j - L| < \varepsilon$ if $j \geq N$

$\Rightarrow |a_{j_k} - L| < \varepsilon$ if $j_k \geq N$

other way: HW

11

Panel 12

Next up is one of the most fundamental thms of analysis:

Thm. (Bolzano - Weierstrass theorem)

Let $\{a_j\}$ be any sequence that is bdd. Then there exists a convergent subsequence!

Ex: $\{\sin(j)\} = \{0, 0.8414, 0.90929, 0.1411, \dots\}$

Bdd by $1 \Rightarrow \exists$ conv. subsequence by BW (easy)

Note: Take any $x \in (0,1)$. There is a subseqn. of $\{\sin(j)\} \to x$! Hard

12