Panel 1

Real Analysis - Last time

Sequence: $f: \mathbb{N} \to \mathbb{R}$, $\{a_j\}_{j=1}^{\infty}$ or $\{a_j\}$

Convergent Sequence: If for every $\varepsilon > 0$ $\exists N = N(\varepsilon)$ s.t. $|a_j - L| < \varepsilon$ $\forall j \geq N$, then $\lim_{j \to \infty} a_j = L$

Theorems

$\{a_n\}$ conv. $\Rightarrow$ bounded (converse is false: $(-1)^j$)

$\{a_n\}$ conv. $\Rightarrow$ limits are unique

$\{a_j\}$ is bounded (bdd) if $|a_j| \leq M$ $\forall j$ for some M

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Panel 2

Monotone Sequence

If $a_j \leq a_{j+1}$ then $\{a_j\}$ is monotone increasing

$a_j \geq a_{j+1}$ then $\{a_j\}$ is monoton decreasing

Ex $\left\{\frac{n}{n+1}\right\}$ mon. incr. or decreasing? Increasing

$a_j \lesseqgtr a_{j+1} \Leftrightarrow a_j - a_{j+1} \lesseqgtr 0 \Leftrightarrow \frac{a_j}{a_{j+1}} \lesseqgtr 1$

$\frac{a_j}{a_{j+1}} = \frac{j/(j+1)}{(j+1)/(j+2)} = \frac{j(j+2)}{(j+1)^2} ? 1$

$a_j - a_{j+1} = \frac{j}{j+1} - \frac{j+1}{j+2} = \frac{j(j+2) - (j+1)^2}{(j+1)(j+2)} = \frac{\cancel{j^2} - \cancel{2j} - \cancel{j^2} - \cancel{2j} - 1}{(\ )(\ )} \leq 0$

$\Rightarrow a_j < a_{j+1}$

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Panel 3

Q: If $\{a_n\}$ converges $\Rightarrow \{a_n\}$ is bdd. ✓

If $\{a_n\}$ is bdd, then $\{a_n\}$ converges ✗

$\{n\}$ If $\{a_n\}$ is monotone (incr), then it converges? ✗

If $\{a_n\}$ converges, then it is monotone? ✗

$\left\{\frac{(-1)^n}{n}\right\}$

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Panel 4

Thm: If $\{a_n\}$ is monotone and bdd, then $\{a_n\}$ converges.

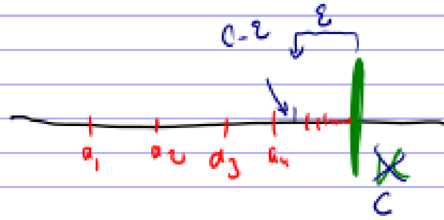
Precisely: (*) a_n increasing and bdd above $\Rightarrow$ conv.

a_n decreasing and bdd below $\Rightarrow$ conv.

$\{a_n\}$ bdd $\to |a_n| \le K$, $\{a_n\}$ bdd above $a_n \le K$

$\{a_n\}$ bdd below $K \le a_n$

(*)



Proof: Say $\{a_j\}$ incr + bdd above. Take $c = \sup(a_j)$. Take any $\varepsilon > 0$.

$\exists N$ s.t. $a_N > c - \varepsilon$

$\Rightarrow a_j > a_N \ \forall j \ge N \Rightarrow a_j > c - \varepsilon \Rightarrow \varepsilon > c - a_j$, $|a_j - c| < \varepsilon \ \forall j \ge N$ q.e.d.

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Panel 5

Ex: $\left\{\frac{n}{n+1}\right\}$ converges.

$\left\{\frac{n}{n+1}\right\}$ is increasing, bdd by 1 $\Rightarrow$ converges!

Ex: $\{a_n\}$ defined recursively as: , $a>0$

$x_0 = a, \quad x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$

Show that $\{a_{n+1}\} \to$

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Panel 6

$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$. Pick $a=2$

$x_0 = 2$

$x_1 = 1.5$

$x_2 = 1.4166$

$x_3 = 1.4142$

$\vdots$

$x_n \to \sqrt{2}$ $(\sqrt{a})$

decreasing

Say I knew $\{x_n\}$ does converge to L. Then

$L = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} =$

$= \lim_{n\to\infty}\left(\frac{1}{2}\left(x_n + \frac{a}{x_n}\right)\right) =$

$L = \frac{1}{2}\left(L + \frac{a}{L}\right)$

$2L = L + a/L$

$L = a/L$

$L^2 = a \quad \Rightarrow L = \sqrt{a}$

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Panel 7

$\underbrace{x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)}$ bounded below by zero

$$x_n - x_{n+1} = x_n - \frac{1}{2}(x_n + a/x_n) = x_n - \frac{1}{2}x_n - \frac{1}{2}a/x_n =$$
$$= \frac{1}{2}(x_n - a/x_n) = \frac{1}{2}\left(\frac{x_n^2 - a}{x_n}\right)$$

Consider $(x_n^2 - a) = \frac{1}{4}\left(x_{n-1} + \frac{a}{x_{n-1}}\right)^2 - a =$

$$= \frac{1}{4}\left(x_{n-1}^2 + 2a + \frac{a^2}{x_{n-1}^2}\right) - a =$$
$$= \frac{1}{4}x_{n-1}^2 - \frac{1}{2}a + \frac{1}{4}\frac{a^2}{x_{n-1}^2} =$$
$$= \frac{1}{4}\left(x_{n-1} - \frac{a}{x_{n-1}}\right)^2 \geq 0$$

$\Rightarrow x_n - x_{n+1} \geq 0 \quad \Rightarrow \{x_n\}$ is decr., bdd below, hence conv.

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Panel 8

Thm: $(1+\frac{1}{n})^n$ converges. to e! What is e?

Define $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ it converges.

Converges because increasing and bdd. above!

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