Panel 1

Real Analysis - Last time

Finished counting: $1, 2, 3, \ldots, \aleph_0, c, 2^c, 2^{2^c}, \ldots$

Partially ordered, Ordered, well-ordered

$P(A)$, $A \subset B$, $(0,1)$ $\mathbb{Q}$, $\mathbb{N}$

Add: Every element except first has predecessor

$\Rightarrow$ Induction works

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Panel 2

$f:(0,1) \to [0,1]$, $f(x) = x$

Since $[.25, .75] \subset (0,1)$,

$g:[0,1] \to [.25,.75]$, $g(x) = \frac{x}{2} + .25$

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Panel 3

$\text{card}(\mathbb{N}\times\mathbb{N}\times\mathbb{N}\times\ldots) \neq \text{card}(\mathbb{N})$

$a \in \mathbb{N}\times\mathbb{N}\times\ldots \Rightarrow a = (a_1, a_2, a_3, a_4, \ldots)$

$D = \{0, 1, \ldots, 9\}$, $a \in D\times D\times\ldots \Rightarrow a = (a_1, a_2, a_3, \ldots)$

Supp. countable $\Rightarrow$ All elements can be listed

$x_1 = (a_1^1, a_2^1, a_3^1, \ldots)$

$x_2 = (a_1^2, a_2^2, a_3^2, \ldots)$

$x_3 = (a_1^3, a_2^3, a_3^3, \ldots)$

$\vdots$

f_i differs from x_i in i-th slot. $\Rightarrow f$ is not in the list.

Define $f = (f_1, f_2, f_3, \ldots)$: $f_i = \begin{cases} 1 & \text{if } a_i^i \neq 1 \\ 0 & \text{if } a_i^i = 1 \end{cases}$

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Panel 4

P_n: $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ all $a_i \in \mathbb{Z}$

Since $\text{card}(\mathbb{Z}) = \text{card}(\mathbb{N})$, $\text{card}(P_n) = \text{card}(\mathbb{N}\cup\mathbb{N}\cup\ldots\cup\mathbb{N}) = \text{card}(A_n)$ $\quad P = \bigcup P_n$

So $\text{card}(P) = \text{card}(A_0 \cup A_1 \cup \cdots \cup A_n \cup \ldots)$
$= \text{card}(\mathbb{N}\cup\mathbb{N}\cup\ldots)$

Since the cardinality of all deg. of polynomials is $\mathbb{N}$, the union of all polynomials w/ integer coefficients is countable.

$\text{card}(P) = \text{card}(\mathbb{N})$

Recall: (1) countable union of ctble sets is ctble. ✓

(2) finite cross-prod. of ctble sets is ctble. ✓

$\mathbb{N}\times\mathbb{N}\times\ldots\times\mathbb{N}$

$p \in P^n$, $f(p) = (a_n, a_{n-1}, a_{n-2}, \ldots, a_1, a_0)$, e.g.

$f(2x^2 - 7x + 1) = (2, -7, 1)$ $\quad f: P^n \to \underbrace{\mathbb{N}\times\ldots\times\mathbb{N}}_{n+1 \text{ times}}$

function bijection $\Rightarrow \text{card}(P^n) = \text{card}(\mathbb{N})$

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Panel 5

$1^3 + 2^3 + 3^3 + \dots n^3 = \frac{1}{4} n^2 (n+1)^2$

for $n = 1$

$\frac{1}{4}(n+1)^2((n+1)+1)^2$

$\boxed{\frac{1}{4}(n+1)^2(n+2)^2}$

$1^3 = 1$

$\frac{1}{4}(1)^2(1+1)^2 = \frac{1}{4}(1)(2)^2 = 1$

let's assume $1^3 + 2^3 + 3^3 + \dots n^3 = \frac{1}{4}(n^2)(n+1)^2$

$\forall n \in \mathbb{N}$

$1^3 + 2^3 + 3^3 + \dots + n^3 + (n+1)^3 = \frac{1}{4}(n^2)(n+1)^2 + (n+1)^3$

$= (n+1)^2\left[\frac{1}{4}n^2 + n + 1\right]$

$= \frac{1}{4}(n+1)^2(n^2 + 4n + 4)$

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Panel 6

Recall induction:

$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$

$n = 1$:

$n \rightsquigarrow n+1$:

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Panel 7

Real Numbers

Done with

Def. A real number x is the equivalence class of all convergent sequences with limit x

Too much work!

Alternate: Dedekind's Cuts   Strange

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Panel 8

Do need: a defining property of $\mathbb{R}$. Need:

Def: Let A be an ordered set, $X \subset A$. An element b is called upper bound for X if $x \le b \ \forall x \in X$.

X is called bounded above.

An element b in A is a least upper bound of X if it there is no other upper bound b' s.t. $b' < b$. We write $b = \sup(X)$ = supremum

Ex: $S = (0,1) \cap \mathbb{Q} \Rightarrow \sup(S) = 1$

$T = \{1, 1.4, 1.41, 1.414, 1.4142, \ldots\} \to \sqrt{2}$

If all we knew was $\mathbb{Q} \Rightarrow$ no sup for this set T

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Panel 9

Thm: There exists an ordered field of numbers s.t.

least upper bd. prop

1. It contains the rational numbers
2. It has the property that every non-empty subset with an upper bound has a least upper bound

We call this $\mathbb{R}$!

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Panel 10

Prop: There is no $x \in \mathbb{Q}$ s.t. $x^2 = 2$

Assume it is $\in \mathbb{Q} \Rightarrow x = \frac{a}{b}$ (fully reduced)

$\Rightarrow \frac{a^2}{b^2} = 2 \Rightarrow \boxed{a^2 = 2b^2}$, a^2 is even

$\Rightarrow a$ is even $\Rightarrow a = 2c$

HW $\sqrt{3} \notin \mathbb{Q}$

$(2c)^2 = 2b^2$

$4c^2 = 2b^2$

$2c^2 = b^2 \Rightarrow b^2$ is even, b is even ↯

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Panel 11

Prop: There is an $x \in \mathbb{R}$ s.t. $x^2 = 2$

Uses lub-prop (least upper bd. property)

Define $S = \{t \in \mathbb{R} : t > 0 \text{ and } t^2 \leq 2\}$

$1 \in S$, $2 \notin S$ $\rightarrow S \neq \emptyset$

Let $s = \sup(S)$ $(S \neq \emptyset)$

Need to show: $s^2 = 2$!

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