Panel 1

Real Analysis – Last Time

Learned how to count

$1, 2, 3, \ldots$, card$(\mathbb{N})$, card$(\mathbb{R})$

Countable sets: $\mathbb{N}, \mathbb{Q}$, cble unions, finite crossproducts

Uncountable sets: $\mathbb{R}$, $(0,1)$, $\mathbb{R}\times\mathbb{R}$, $\mathbb{C}$

Continuum Hypothesis:

✓

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Panel 2

Cantor - Bernstein Theorem (Schröder)

If there is $f: A \to B$ injection,
and there is $g: B \to A$ injection — difficult proof
$\Rightarrow \exists\, h: A \to B$ bijection, i.e. card$(A)$ = card$(B)$

Ex: card$((0,1))$ = card$([0,1])$

$f: (0,1) \to [0,1]$, $f(x) = x$

$g: [0,1] \to (0,1)$

Ex: card$((0,1))$ = card$((0,1)\times(0,1))$ $\quad 0.x_1y_1x_2y_2x_3y_3\ldots$

$f: (0,1) \to (0,1)\times(0,1)$: $f(x) = (x, 0)$

$g: (0,1)\times(0,1) \to (0,1)$, $g(x,y) = g(0.x_1x_2x_3\ldots, 0.y_1y_2y_3\ldots) =$

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Panel 3

Power set $P(S)$: set of all subsets $\{a,b,c,..\}$ $\{A,B,C,..\}$

Thm: $\text{card}(P(S)) > \text{card}(S)$

Assume $\text{card}(P(S)) = \text{card}(S)$, i.e. $\exists g: S \to P(S)$

s.t. $g(a) = A$, $g(b) = B$, $g(c) = C$ ...

Define

$$X = \{ s \in S : \{s\} \cap f(s) = \emptyset \}$$

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Panel 4

Ex: $S = \{1,2,3\}$, $S = \{\emptyset, S, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$

$g: S \to P(S)$ could be

$g(1) = \{3\}$ $\quad g(2) = \{1,2,3\}$ $\quad g(3) = \{1,2\}$

$X = \{ \{s\} \cap g(s) = \emptyset \}$

$\{1\} \cap g(1) = \emptyset$ $\quad \{2\} \cap g(2) \neq \emptyset$ $\quad \{3\} \cap g(3) = \emptyset$

$X = \{1,3\}$ Note: no element from $S$ is associated with $X$

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Panel 5

So far: $S = \{a, b, \ldots\}$, $P(S) = \{A, B, \ldots\}$, $g: S \to P(S)$

$X = \{s \in S : \{s\} \cap g(s) = \emptyset\}$

Since $g: S \to P(S)$, $X \subset P(S)$ $\Rightarrow \exists\, x$ s.t. $g(x) = X$

case 1. $x \in X$: $\Rightarrow \{x\} \cap g(x) \neq \emptyset$ $\Rightarrow x \notin X$

case 2: $x \notin X$ $\Rightarrow \{x\} \cap X = \emptyset$ $\Rightarrow x \in X$

Thus, there is no $x$ with $g(x) = X$ !

$\Rightarrow \text{card}(P(S)) \neq \text{card}(S)$

$\Rightarrow \text{card}(P(S)) > \text{card}(S)$

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Panel 6

We now have Hierarchy of Infinity

$1, 2, 3, \ldots, \text{card}(\mathbb{N}), \text{card}(\mathbb{R}),$ $\text{card}(P(\mathbb{R})), \text{card}(P(P(\mathbb{R})))$

real analysis $2^c = 2^{2^{\aleph_0}}$

Can show: $\text{card}(P(\mathbb{N})) = \text{card}(\mathbb{R})$

$\text{card}(\mathbb{N}) = \aleph_0$ (aleph-null) $\text{card}(P(\mathbb{N})) = \text{card}(\mathbb{R})$

$\text{card}(\mathbb{R}) = c$ $2^{\aleph_0} = c$

$\aleph_0 + \aleph_0 = \aleph_0$

ordinal #'s

$\aleph_0 + c = c$

$c + c = c$

$\aleph_0 - \aleph_0 =$ undefined

ENOUGH ALREADY

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Panel 7

# A closer look at Induction

**Def:** A set $S$ is partially ordered if there is a relation $<$ s.t.

a) $a < a$ (reflex)

b) $a < b$ and $b < c \Rightarrow a < c$ (trans)

c) $a < b$ and $b < a \Rightarrow a = b$ (anti-symmetri)

$S$ is ordered if it is partially ordered and all elements can be compared. $S$ is called well-ordered if it is ordered and if $A \subset S$, $A \neq \emptyset$, $A$ has smallest element.

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Panel 8

Partially ordered? ordered? well-ordered?

**Ex:** a) $\mathbb{N}, \leq$ well-ordered

b) $\mathbb{Q}, \leq$ ordered, not well-ordered

c) $S$ any set. Define $a < b$ if $a = b$ part. ordered

d) $S$ any set, $P(S)$ the power set. Define $A < B$ if $A \subset B$ $\Rightarrow$ part. order

e) Consider $\mathbb{N}$ and define a relation $<$ by:

- if $n, m$ are even then $n < m$ if $n \leq m$
- if $n, m$ are odd then $n < m$ if $n \leq m$
- if $n$ even, $m$ odd then $n < m$

$\{2, 4, 6, 8, \ldots, 1, 3, 5, \ldots\}$ well-ordered

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Panel 9

Thm: Induction Principle

Let $S$ be well-orded set such that every element except the smallest has an immediate predecessor.

Then: if $Q$ is a property s.t.

(a) smallest element in $S$ has property $Q$

(b) If $s$ has property $Q$ then successor of $s$ has $Q$

then $Q$ holds for all $s \in S$.

Proof: Say NOT: if smallest element has $Q$, if $s$ has $Q$ then $s'$ (successor of $s$) has $Q$ but $Q$ does not apply to all of $S$

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Panel 10

Let $X = \{s : Q \text{ does not hold}\}$

$X \neq \emptyset$. $S$ is well-ordered $\Rightarrow$ $X$ has smallest element $e$

$e$ is not the first (by assumption)

$e$ has imm. predecessor $\hat{e}$

$Q$ holds for $\hat{e}$. Know that if $Q$ is true for $s$ then it is true for $s'$

$\Rightarrow Q$ must be true for $\hat{e}' = e$

↯

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Panel 11

$\mathbb{N}$ with relation $<$ s.t.

Well-ordered

$n, m$ odd $\Rightarrow$ $n < m$ if $n \leq m$

1 has no predecessor.

$n, m$ even $\Rightarrow$ $n < m$ if $n \leq m$

$n$ even, $m$ odd $\Rightarrow$ $n < m$

Prove that all numbers are even: By induction, using $\mathbb{N}$ with this ordering.

HW

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Panel 12

Strange Induction Proof.

All birds have same color

?

Proof: $n = 1$: obviously true

Suppose true for $n$ birds;

grab $(n+1)$ birds, take one out.

$n$ remaining birds have same color, say red.

Take out one more red bird, through original back in.

$\Rightarrow n \Rightarrow$ they must all be red. Throw left-over in

$\Rightarrow n+1$ - red birds.

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Panel 13

## Old-fashioned Induction

(1) $(a+b)^n \geq a^n + b^n \quad \forall a, b > 0$

(2) $1+2+3+\ldots+n = \frac{n(n+1)}{3}$

(3) $1^2+2^2+3^2+4^2+\ldots+n^2 = \frac{1}{6} n(n+2)(2n+1)$

(4) $1^3+2^3+3^3+4^3+\ldots+n^3 = \frac{1}{4} n^2(n+1)^2$

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