Panel 1

Last Time

Defined $\mathbb{N}$ (via Peano Axioms), $\mathbb{Z}$ and $\mathbb{Q}$

Cardinality: card(A) = card(B) means:
$f: A \to B$ bijection

Questions:

card(even) = card(odd)
card(even) = card($\mathbb{N}$)
card($\mathbb{Z}$) = card($\mathbb{N}$)
card($\mathbb{Q}$) = card($\mathbb{N}$) ?
card($\mathbb{R}$) $\overset{?}{=}$ card($\mathbb{N}$)

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Panel 2

Thm: card(even) = card(odd)

$f: E \to O$

$f: E \to \mathbb{N}$ $\quad h = g^{-1} \circ f$

$g: O \to \mathbb{N}$

$f(x) = x - 1$ $\quad y \in O$, $x = y+1 \in E \Rightarrow f(x) = y+1-1 = y$

card($\mathbb{N}$) = card($\mathbb{Z}$)

$$f: \mathbb{N} \to \mathbb{Z}, \quad f(x) = \begin{cases} \frac{x}{2} & \text{, if } x \text{ even} \\ -\frac{x-1}{2} & \text{, if } x \text{ odd} \end{cases}$$

Onto: $y \in \mathbb{Z}$, $y > 0$. Take $x = 2y$, $f(x) = \frac{x}{2} = y$

$y < 0$ Take $x = 1 - 2y$ odd, $f(x) = -\frac{1-2y-1}{2} = y$

1-1 because linear.

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Panel 3

Def. A set A is called countable if
$\text{card}(A) = \text{card}(\mathbb{N})$

Note: Countable means: I can count it!

Thm. (1) Every subset of a countable set is countable

(2) The set of ordered pairs $\mathbb{N}\times\mathbb{N}$ is countable

(3) The countable union of ctble sets is countable

(4) The finite crossproduct of ctble sets is countable

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Panel 4

$\mathbb{N}\times\mathbb{N}$ is countable.

Proof. $(1,1),(1,2),(1,3)\ldots(2,1),(2,2),(2,3),\ldots(3,1),(3,2),(3,3)\ldots$

$$\begin{array}{ccccc}
(1,1) & (1,2) & (1,3) & (1,4) & \ldots \\
(2,1) & (2,2) & (2,3) & (2,4) & \ldots \\
(3,1) & (3,2) & (3,3) & (3,4) & \ldots \\
(4,1) & (4,2) & (4,3) & (4,4) & \ldots \\
\vdots & & \vdots & \vdots &
\end{array}$$

This counts them all $\Rightarrow$ $\mathbb{N}\times\mathbb{N}$ is countable via this "counting" bijection.

q.e.d

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Panel 5

Countable union of countable sets is countable

$$U_1 = \{\overset{1}{u_{11}}, \overset{2}{u_{12}}, \overset{6}{u_{13}}, u_{14}, \ldots\}$$

$$U_2 = \{\overset{3}{u_{21}}, \overset{5}{u_{22}}, u_{23} \ldots\}$$

$$U_3 = \{\overset{4}{u_{31}}, u_{32}, u_{33}, \ldots\}$$

$$U_4 \quad \vdots$$

$$\vdots$$

Diagonal counting applies again!

q.e.d.

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Panel 6

Finite cross product of countable sets is countable

$$(\mathbb{N} \times \mathbb{N}) \times \mathbb{N} \times \mathbb{N} \times \ldots \times \mathbb{N}$$

$$(A_1 \times \mathbb{N}) \times \mathbb{N} \times \ldots \times \mathbb{N}$$

$$(A_2 \times \mathbb{N}) \times \ldots \times \mathbb{N}$$

$$\vdots$$

done after finite steps $\Rightarrow$ countable!

HW What about countable crossproduct of countable sets

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Panel 7

What is card($\mathbb{Q}$) ?

Proof: Take $r \in \mathbb{Q}$, $r = p/q$, $q > 0$

Define height of r: $h = |p| + q$

Ex: Height of $5/2$ is 7, and of $-4/3$ is 7

$h = 1$: $\frac{0}{1}$

$h = 2$: $-1/1$ $\frac{0}{2}$ $\frac{1}{1}$

$h = 3$: $-2/1$, $-1/2$, $0/3$, $1/2$, $2/1$

$h = 4$: $-3/1$ $-2/2$ $-1/3$ $\frac{0}{4}$ $1/3$ $2/2$ $3/1$

This set is therefore countable, thus $\mathbb{Q}$ which is a subset, is countable as well.

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Panel 8

Are there any sets that are not countable?

Thm: $(0,1) \subset \mathbb{R}$ is not countable (uncountable)

Try prove by contradiction: assume $(0,1)$ was countable

$\Rightarrow$ $x_1 = 0.x_1^1 x_2^1 x_3^1 x_4^1 \ldots$ all in $(0,1)$

$x_2 = 0.x_1^2 x_2^2 x_3^2 x_4^2 \ldots$

$x_3 = 0.x_1^3 x_2^3 x_3^3 x_4^3 \ldots$ $\Rightarrow t = 0.010\ldots$

$x_4 = 0.x_1^4 x_2^4 x_3^4 x_4^4 \ldots$ $t \in (0,1)$ ✓

Define $t = 0.t_1 t_2 t_3 t_4 \ldots \in (0,1)$ where

$t_1 = \begin{cases} 0 & \text{if } x_1^1 \neq 0 \\ 1 & \text{if } x_1^1 = 0 \end{cases}$, $t_2 = \begin{cases} 0 & \text{if } x_2^2 \neq 0 \\ 1 & \text{if } x_2^2 = 0 \end{cases}$ ...

$t \neq x_1$ in 1st pos

$t \neq x_2$ in 2nd pos

$t \neq x_3$ in 3rd pos

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Panel 9

Thus, t is not part of list $x_1, x_2, \ldots$

Thus, that list is not all of $(0,1)$

↯

Therefore $(0,1) \subset \mathbb{R}$ is uncountable!

HW $(-1,1)$ is uncountable $(f(x) = 2x-1)$

(a,b) is uncountable

$\mathbb{R}$ is uncountable, in part. $\text{card}(\mathbb{R}) = \text{card}(0,1)$

Hint: $h(x) = \arctan(x)$

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Panel 10

Is the countable crossproduct of countable sets countable or uncountable?

HW <u>no hints</u>

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Panel 11

So far we count:

1, 2, 3, ... card(ℕ), card(ℝ)

⇒ 2 Questions:

1.) Is there A s.t. $\text{card}(\mathbb{N}) < \text{card}(A) < \text{card}(\mathbb{R})$

2.) Is there A s.t. $\text{card}(A) > \text{card}(\mathbb{R})$

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Panel 12

Continuum Hypothesis: Is there a set A s.t. $\text{card}(\mathbb{N}) < \text{card}(A) < \text{card}(\mathbb{R})$?

There are things in Math (set of logical rules that can at least explain the $(\mathbb{Z}, +, \cdot)$) that can not be proven or disproven. So they are simultaneous true and false!!.

Can prove this! ⇒ Gödel, Escher, Bach

Even if you arbitrarily say "it's true" and add it as an axiom ⇒ there is another unprovable statement

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Panel 13

Paradox: The barber of Seville shaves all men that don't shave themselves.

Who shaves the barber?

$\mathbb{R}$ is sometimes called "the continuum"

$\mathbb{R}$ has no holes

$\mathbb{Q}$

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Panel 14

Are there sets S s.t. $\text{card}(S) > \text{card}(\mathbb{R})$

Note: $\text{card}(\mathbb{R}\times\mathbb{R}) = \text{card}(\mathbb{R})$

$\text{card}(\mathbb{R}^n) = \text{card}(\mathbb{R})$

$\mathbb{R}^\infty$?

Def: S any set. Let $P(S)$ = set of all subsets of S, called the power set of S.

$S = \{0,1\} \Rightarrow P(S) = \{\emptyset, S, \{0\}, \{1\}\}$

$S = \{0,1,2\} \Rightarrow P(S) = \{\emptyset, S, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}\}$

HW: $P(\{0,1,2,3\}) \Rightarrow \text{card}(P(S)) = ?$

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Panel 15

Q2: Is there set A such that card(A) > card($\mathbb{R}$)?

Def: The power set of A is the set of all subsets of A. That new set is called power set of A, or P(A).

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