Panel 1

Proof by contrapositive

If x isn't divible by 3, then x^2 isn't divible by 3.

Case 3r1

$x = 3k+1, \quad k \in \mathbb{Z}$

$x^2 = (3k+1)(3k+1)$

$= 9k^2 + 6k + 1$

$= 3(3k^2 + 2k) + 1$.

Since $3k^2 + 2k \in \mathbb{Z}$,

$x^2 = 3k+1$

not divible by 3

Case 3r2

$x = 3k+2, \quad k \in \mathbb{Z}$.

$x^2 = (3k+2)(3k+2)$

$= 9k^2 + 12k + 4$

$= 9k^2 + 12k + 3 + 1$

$= 3\underbrace{(3k^2 + 4k + 1)}_{\in \mathbb{Z}} + 1$

x^2 is not divble by 3

Q.E.D.

1

arena

Panel 2

Prove $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$\rightarrow$

Let x be in $A \cap (B \cup C)$

x is in A and $(B \cup C)$

x is in B or C

CASE 1: Let x be in B

then x is in $(A \cap B)$ so x is in $(A \cap B) \cup (A \cap C)$ ✓

CASE 2: Let x be in C then x is in $(A \cap C)$ so x is in $(A \cap B) \cup (A \cap C)$ ✓

2

rashdan

Panel 3

Let x be in $(A \cap B) \cup (A \cap C)$
x is in $(A \cap B)$ or $(A \cap C)$
CASE 1: Let x be in $(A \cap B)$. So x is in A and B So x is in A and $(B \cup C)$ so x is in $A \cap (B \cup C)$

CASE 2: Let x be in $(A \cap C)$ then x is in A and C so x is in A and $(B \cup C)$ so $A \cap (B \cup C)$ ✓
q.e.d.

3

Panel 4

Relations: A, B are two sets. A relation is collection of ordered pairs (a, b), $a \in A$, $b \in B$. We say that $a \sim b$ (a is related to b)

Function: A relation that relates for every $a \in A$ exactly one $b \in B$.
A is domain, B = range

Ex: $A = \{1, 2, 3, 4\}$, $B = \{19, 7, 234\}$

$\underbrace{\sigma: 1 \sim 234,\ 2 \sim 7,\ 3 \sim 19,\ 4 \sim 234}_{\text{function}}$ $, 2 \sim 19$

no longer

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Panel 5

$f: \mathbb{R} \to \mathbb{R},\ f(x) = x^2$

<u>Def</u> $f: \mathbb{R} \to \mathbb{R}$ be a function. We define the image of f, or image of domain, as:

$$\text{imag}(f) = \{ b \in B \text{ s.t. there is } a \in A \text{ with } f(a) = b \}$$

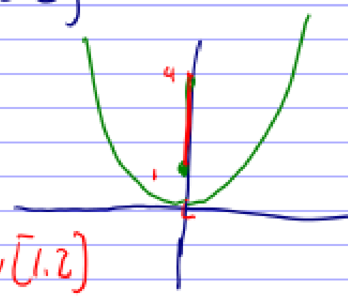
The preimage of $C \subset A$ is:

$$f^{-1}(C) = \{ a \in A : f(a) \in C \}$$

$f(x) = x^2$ $\quad \text{imag}(f) = [0, \infty)$

$\text{imag}([0,2]) = [0,4]$

$f^{-1}([1,4]) = [-2,-1] \cup [1,2]$

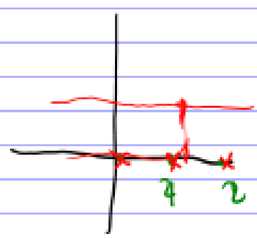


5

Panel 6

<u>Dirichlet Function</u>

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \text{ (rational)} \\ 1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \text{ (irrational)} \end{cases}$$



no graph to see

$\text{imag}(f) = \{0,1\}$

$\text{imag}([0,2]) = \{0,1\}$

$f^{-1}([-\tfrac{1}{2}, \tfrac{1}{2}]) = \mathbb{Q}$

$f^{-1}([\tfrac{1}{2}, 2]) = \mathbb{R} \setminus \mathbb{Q}$

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Panel 7

Def. A function $f: A \to B$ is
injective (1-1) if whenever $f(a) = f(b) \Rightarrow a = b$
surjective (onto) if $imag(f) = B$
bijective if injective + surjective

$g: \mathbb{R} \to \mathbb{R}$, $g(x) = \sin(x)$ not 1-1 $(f(0) = f(\pi) = 0)$
not surjective $imag(f) = [-1,1]$

Note: Not every function is 1-1, but every function can be made onto by setting $B = imag(f)$

$\hat{g}: \mathbb{R} \to [-1,1]$, $\hat{g}(x) = \sin(x)$

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Panel 8

Equivalence Relations: A relation r s.t.
1. Reflexive : $a \sim a \; \forall a \in A$
2. Symmetric : $a \sim b$ then $b \sim a$
3. Transitive: $a \sim b$ and $b \sim c \Rightarrow a \sim c$

Ex: $A = \{1,2,3,4\}$, $B = \{a,b,c\}$

$r: \{(a,a),(b,b),(a,b),(b,a)\} \cup \{(c,c)\}$ IS

$s: 1\sim1, 2\sim2, 3\sim3, 4\sim4, 1\sim4, 4\sim1, 2\sim4, 4\sim2$

$1\sim4$, $4\sim2$ but not $1\sim2$ Not

8

Panel 9

Thm: Let r be an equiv. relation on A. Then A can be written as

$$A = \bigcup_\alpha A_\alpha$$

(1) If $a, b \in A$ then $a \sim b$ iff $a, b \in A_\alpha$

(2) A_α are non-empty and disjoint

A_α are called equiv. classes

Ex: Take $\mathbb{Z}$, r is relation: $x \sim y$ if $y - x$ div. by 2.

Is it equiv relation? $x \sim x$: $x - x = 0/2$ ✓

If $x - y = 2n$ then $y - x = 2(-n)$

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Panel 10

$x \sim y$: $y - x = 2n$

$y \sim z$: $z - y = 2m$

$\Rightarrow z - x = z - y + y - x = 2(n+m)$ trans ✓

Say A_1 is class cont. 1

$A_1 = \{1, 3, 5, 7, 9, 11, \ldots\} = [(1)] = O$

$A_2 = \{2, 4, 6, \ldots\} = [(2)] = E$

$[(2)] = [(4)] = [(126)] = [(0)]$

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Panel 11

$[(0)] + [(0)] = [(0)]$

$[(0)] + [(1)] = [(1)]$

$[(1)] + [(1)] = [(0)]$

Binary #

11

Panel 12

Def. The Integers

Define a relation on $\mathbb{N} \times \mathbb{N}$ by saying $(a,b) \sim (a',b')$ are related if $a + b' = a' + b$

If $[(a,b)]$ and $[(a',b')]$ are equiv. classes, then

$$[(a,b)] + [(a',b')] = [(a+a', b+b')]$$

$$[(a,b)] \cdot [(a',b')] = [(ab' + ba', aa' + bb')]$$

$(1,3) \sim (2,4) \sim (3,5) \sim (121,123) = [(1,3)]$

$(0,0) \sim (1,1) \sim (2,2) \ldots \qquad [(0,0)]$

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Panel 13

$[(7,2)] = \{(7,2), (8,3), \ldots\}$

$[(1,2)] + [(4,6)] = [(5,8)]$

$[(1,2)] \cdot [(4,6)] = [(1\cdot 6 + 2\cdot 4, 1\cdot 4 + 2\cdot 6)] =$

$[(14,16)]$

Who, really, is

$[(7,5)] \approx 2$

$[(5,7)] \approx -2$

$3 \cdot (-2) =$

$[(4,1)] \cdot [(1,3)] =$

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