Math 2511: Calc III - Practice Exam 3

1. State the meaning or definitions of the following terms:
   a) vector field, conservative vector field, potential function of a vector field, volume, length of a curve, work, surface area
   b) curl and divergence of a vector field F, gradient of a function
   c) $\iint_R dA$ or $\iint_R f(x,y)dA$ or $\iiint_Q f(x,y,z)dV$
   d) $\int_C ds$ or $\int_C f(x,y)ds$ or $\int_C f(x,y)dx$ or $\int_C f(x,y)dy$
   e) $\int_C \vec{F}\cdot d\vec{r}$
   f) $\iint_S g(x,y,z)\cdot dS$
   g) $\int_C M(x,y,z)dx + N(x,y,z)dy + P(x,y,z)dz$
   h) What does it mean when a "line integral is independent of the path"?

   see notes or book

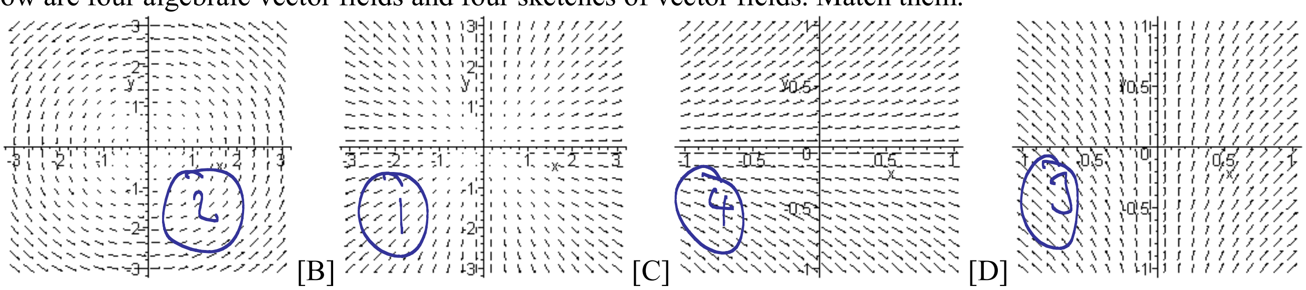
   i) State the Fundamental Theorem of Line Integrals. Make sure to know when it applies, and when it helps.]

   $\int_C \vec{F}\,d\vec{r} = f(B) - f(A)$. C curve from A to B, f potential of F. Applies if $\vec{F}$ is conservative

   j) State Green's Theorem. Make sure to know when it applies, and in what situation it helps.

   $\oint_C M\,dx + N\,dy = \iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,dA$. C closed curve, pos. oriented, R inside of C

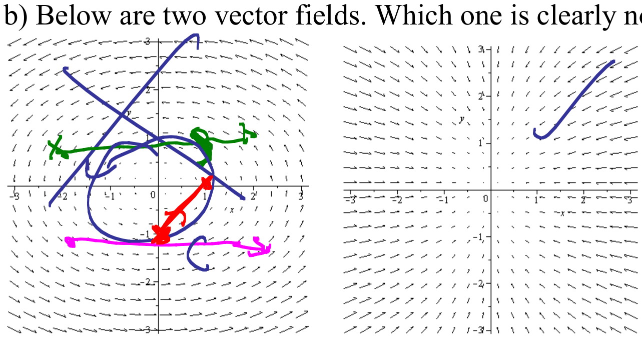
2. Below are four algebraic vector fields and four sketches of vector fields. Match them.



[A] [B] [C] [D]

(1) $F(x,y) = \langle x, y\rangle$, (2) $F(x,y) = \langle -y, x\rangle$, (3) $F(x,y) = \langle x, 1\rangle$, (4) $F(x,y) = \langle 1, y\rangle$

b) Below are two vector fields. Which one is clearly not conservative, and why?



The first because $\oint_C F\,dr \neq 0$

c) Say in the left vector field above you integrate over a straight line from (0,-1) to (1,0). Is the integral positive, negative, or zero? pos (at work)

How about if you integrate from (-2,1) to (2,1)? Neg. (cosh work)

How about from (-2,-1) to (2,-1)? positive

3. Are the following statements true or false:
   a) If the divergence of a vector is zero, the vector field is conservative. F
   b) If $F(x,y,z)$ is a conservative vector field then $curl(F)=0$ T
   c) If a line integral is independent of the path, then $\int_C F\cdot dr = 0$ for every path C F
   d) If a vector field is conservative then $\int_C F\cdot dr = 0$ for every *closed* path C T
   e) $\iint_R dA$ denotes the surface area of the region R F
   f) $\iint_R f(x,y)dA$ denotes the volume of the region under the surface *f(x,y)* and over R, if *f* is positive. T
   g) Can you apply the Fundamental Theorem of line integrals for the function $f(x,y,z) = xy\sin(z)\cos(x^2+y^2)$? F
      applies to vector fields
   h) Can you apply the Fundamental Theorem of line integrals for the vector field T
      $F(x,y) =< 6xy^2 - 3x^2, 6x^2y + 3y^2 - 7 >$? is conservative
   i) Can you apply Green's theorem for a curve C, which is a straight line from (0,0,0) to (1,2,3)? F requires closed curve
   j) Can you apply Green's theorem for a closed curve C given by $r(t) =< x(t), y(t), z(t) >$ F must be 2D

4. Suppose that $F(x,y,z) =< x^3y^2z, x^2z, x^2y >$ is some vector field.
   a) Find *div(F)*

   $F_x + F_y + F_z = 3x^2y^2z + 0 + 0$

   b) Find *curl(F)*

   $\begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ x^3y^2z & x^2z & x^2y \end{vmatrix} = \langle x^2 - x^2, -(2xy - x^3y^2), 2xz - 2x^3yz \rangle$

   c) Find *curl(curl(F))*

   do curl again ...

   d) Find *div(curl(F))*

   $div(0, x^3y^2 - 2xy, 2xz - 2x^3yz) = 0 + \cancel{2x^3y} - 2x + 2x - \cancel{2x^3y} = 0$

   e) *grad.*, *div.*, and *curl* of the vector field if appropriate for $< x^2, y^2, z^2 >$

   $div(x^2, y^2, z^2) = 2(x+y+z)$, $\begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ x^2 & y^2 & z^2 \end{vmatrix} = \langle 0,0,0 \rangle$

   f) *grad.*, *div.*, and *curl* of the vector field if appropriate for $< \cos(y) + y\cos(x), \sin(x) - x\sin(y), xyz >$

   div and curl are appropriate, left as HW

g) *grad.*, *div.*, and *curl* of the vector field if appropriate for $f(x,y,z) = z\ln(x^2+y^2)$

$$\text{grad}\left(z\ln(x^2+y^2)\right) = \left\langle \frac{2xz}{x^2+y^2}, \frac{2yz}{x^2+y^2}, \ln(x^2+y^2) \right\rangle$$

5. Decide which of the following vector fields are conservative. If a vector is conservative, find its potential function

a) $F(x,y) = \langle 2xy, x^2 \rangle$

conservative, $f = x^2 y + C$

b) $F(x,y) = \langle e^x \cos(y), e^x \sin(y) \rangle$

not conservative

c) $F(x,y,z) = \langle \sin(y), -x\cos y, 1 \rangle$

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Curl[{Sin[y], -x Cos[y], 1}, {x, y, z}]
{0, 0, -2 Cos[y]}
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Not

d) $F(x,y,z) = \langle 2xy, x^2+z^2, 2zy \rangle$

$f = x^2 y + z^2 y$; $f_x = 2xy$, $f_y = x^2+z^2$, $f_z = 2yz$ ✓

e) $F(x,y) = \langle 6xy^2 - 3x^2, 6x^2 y + 3y^2 - 7 \rangle$

conservative ; $f = 3x^2y^2 + x^3 + y^3 - 7y + C$

f) $F(x,y) = \langle -2y^3\sin(2x), 3y^2(1+\cos(2x)) \rangle$

not

g) $F(x,y,z) = \langle 4xy + z, 2x^2 + 6y, 2z \rangle$

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Curl[{4 x y + z, 2 x^2 + 6 y, 2 z}, {x, y, z}]
{0, 1, 0}
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Not

h) $F(x,y,z) = \langle 4xy + z^2, 2x^2 + 6yz, 2xz \rangle$

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Curl[{4 x y + z^2, 2 x^2 + 6 y z, 2 x z}, {x, y, z}]
{-6 y, 0, 0}
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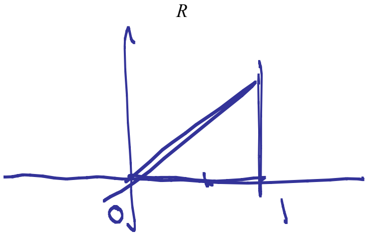
Not

6. Evaluate the following integrals:

a) $\iint_R \cos(x^2)\,dA$ where R is the triangular region bounded by y = 0, y = x, and x = 1

$$\int_0^1 \int_0^x \cos(x^2)\,dy\,dx = \int_0^1 x\cos(x^2)\,dx = \frac{1}{2}\sin(x^2)\Big|_0^1 = \frac{1}{2}\sin(1)$$



b) $\int_0^1 \int_1^{2y} x^2 y^3 \, dxdy$ one computes $= \frac{25}{54}$

$r' = \langle 2t, 1 \rangle$

c) $\int_C ds$, where C is the curve given by $r(t) = \langle t^2, 1+t \rangle$, $0 \le t \le 2$ (you might want to use Maple at some point)

$$\int_0^2 \sqrt{4t^2+1}\, dt =$$

one computes

d) $\int_C x^2 y^3 dx$, where C is the curve given by $r(t) = \langle t^2, t^3 \rangle$, $0 \le t \le 2$

$$\int_0^2 (t^2)^2 (t^3)^3 \, 2t \, dt =$$ one computes

e) $\int_C x^2 - y + 3ds$ where C is the circle $r(t) = \langle 2\cos(t), 2\sin(t) \rangle$, $0 \le t \le \pi$

$$\int_0^\pi \left((2\cos(t))^2 - (2\sin(t)) + 3 \right) \sqrt{4(\cos^2 + \sin^2)}\, dt =$$ computes

f) $\int_C x^2 - y + 3zds$ where C is a line segment given by $r(t) = \langle t, 2t, 3t \rangle$, $0 \le t \le 1$

$$\int_0^1 (t)^2 - (2t) + 3(3t) \cdot \sqrt{1+4+9}\, dt =$$ computes

g) $\int_C F \cdot dr$ where $F(x,y) = \langle y, x^2 \rangle$ and C is the curve given by $r(t) = \langle 4-t, 4t-t^2 \rangle$, $0 \le t \le 3$

$$\int_C y\,dx + x^2 dy = \int (4t-t^2)(-1)\,dt + (4-t)^2 (4-2t)\,dt =$$ computes

h) $\int_C F \cdot dr$ where $F(x,y) = \langle yz, x^2, zy \rangle$ and C is the curve given by $r(t) = \langle 1-t, 3t, 2-t^2 \rangle$, $1 \le t \le 3$

$$\int_C yz\,dx + x^2 dy + zy\,dz = \int_1^3 (3t)(2-t^2)(-1)\,dt + (1-t)^2\, 3\,dt + (2-t^2)(3t)(-2t)\,dt =$$ computes

i) $\int_C ydx + x^2 dy$ where C is a parabolic arc given by $r(t) = \langle t, 1-t^2 \rangle$, $-1 \le t \le 1$

$$\int_{-1}^1 (1-t^2)\,dx + (t)^2 (-2t)\,dt =$$ computes

j) Find the surface integral $\iint_S x - 2y + z\,dS$, where S is the surface $z = 10 - 2x + 2y$ such that x is between 0 and 2 and y is between 0 and 4.

$= f(x,y)$ so $f_x = -2$, $f_y = 2$,

$dS = \sqrt{1+4+4}\,dA$

$$\int_0^2\int_0^4 x - 2y + (10 - 2x + 2y)\sqrt{1+4+4}\,dy\,dx = \text{compute}$$

l) What is the surface area of the plane $6x + 4y + 2z = 8$, where $0 \le x \le 2$ and $0 \le y \le 1$.

$z = 4 - 3x - 2y$, $f_x = -3$, $f_y = -2$, $dS = \sqrt{1+9+4}\,dA = \sqrt{15}\,dA$ $\Rightarrow S = \iint_S dS = \sqrt{15}\int_0^2\int_0^1 dy\,dx = 2\sqrt{15}$

k) $\iint_S (x + z)dS$ where S is the first-octant portion of the cylinder $y^2 + z^2 = 9$ between x = 0 and x = 4

$z = \pm\sqrt{9-y^2}$, $f_x = 0$, $f_y = -y/\sqrt{9-y^2}$, $1 + f_x^2 + f_y^2 = 1 + \frac{y^2}{9-y^2} = \frac{9-y^2+y^2}{9-y^2} = \frac{9}{9-y^2}$

$$\Rightarrow \iint_R \left(x + \sqrt{9-y^2}\right)\cdot\sqrt{\frac{9}{9-y^2}}\,dy\,dx = \int_0^4\int_0^3 \left(x+\sqrt{9-y^2}\right)\frac{3}{\sqrt{9-y^2}}\,dy\,dx = \text{compute}$$

7. For some of the following line integrals there may be short-cut you can use to simplify your computations (but justify your shortcut by quoting the appropriate theorem)

a) $\int_C F \cdot dr$ where $F(x,y) = \langle e^x\cos(y), -e^x\sin(y)\rangle$ and C is the curve $r(t) = \langle 2\cos(t), 2\sin(t)\rangle$, $0 \le t \le 2\pi$

closed curve so use Green: $= \iint (-e^x\sin(y) - (-e^x\sin(y)))\,dA = 0$

b) $\int_C 2xyz\,dx + x^2z\,dy + x^2y\,dz$ where C is some smooth curve from (0,0,0) to (1,4,3)

use potential. $f(x,y,z) = x^2yz + C$ (guessed) $= x^2yz\Big|_{(0,0,0)}^{(1,4,3)} = 1\cdot4\cdot3 = 12$

c) $\int_C F \cdot dr$ where $F(x,y) = \langle y^3 + 1, 3xy^2 + 1\rangle$ and C is the upper half of the unit circle, from (1,0) to (-1,0)

conservative with potential $y^3x + x + y + C$ $= y^3x + x + y\Big|_{(1,0)}^{(-1,0)} = -1 - 1 = -2$

d) $\int_C F \cdot dr$ where $F(x,y) = \langle y^3x, 3xy^2\rangle$ and C is the line segment from (-1,0) to (2,3).

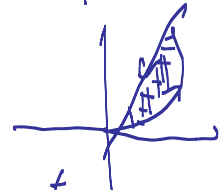
No shortcut: $r(t) = \langle -1,0\rangle + t\langle 3,3\rangle = \langle -1+3t, 3t\rangle$, $t \in [0,1]$

$$\to \int_C y^3x\,dx + 3xy^2\,dy = \int_0^1 (3t)^3(-1+3t)\cdot 3\,dt + 3(-1+3t)(3t)^2\cdot 3\,dt = \text{compute}$$

e) $\int_C y^3dx + (x^3 + 3xy^2)dy$ where C is the path from (0,0) to (1,1) along the graph of $y = x^3$ and from (1,1) to (0,0) along the graph of $y = x$.

~~$r(t) = \langle t, t^3\rangle$, $t \in [1,0]$~~

closed curve, so Green's thm: $= \iint_R (3x^2 + \cancel{3y^2}) - \cancel{3y^2}\,dA = 3\iint_R x^2\,dA = 3\int_0^1\int_{x^3}^{x} x^3\,dy\,dx = \text{compute}$



8. Green's Theorem

a) Use Green's theorem to find $\int_C F \cdot dr$ where $F(x,y) =< y^3, x^3 + 3xy^2 >$ and C is the circle with radius 3, oriented counter-clockwise (You may need the double-angle formula for cos somewhere during your computations, or use Maple)

Closed curve: $\oint_C \vec{F}\, d\vec{r} = \iint_{\text{Disk}} \underbrace{3x^2 + \cancel{3y^2}}_{\frac{\partial N}{\partial x}} - \cancel{3y^2} = \int\int 3x^2\, dr\, d\theta =$

$= \int\int 3\left(3^2\cos^2\theta\right) r\, dr\, d\theta =$ calculator

b) Evaluate $\iint_R dA$ where R is the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ by using a vector field $F(x,y) =< -\frac{y}{2}, \frac{x}{2} >$ and the boundary C of the ellipse R. Note that we did this in class, it is a very special application of Green's theorem.

$A = \oint_C$

see extra credit section

9. Evaluate the following integrals. You can use any theorem that's appropriate:

c) $\int_C 2xyzdx + x^2zdy + x^2ydz$ where C is a smooth curve from (0,0,0) to (1,4,3)

see previous examples

d) $\int_C ydx + 2xdy$ where C is the boundary of the square with vertices (0,0), (0,2), (2,0), and (2,2)

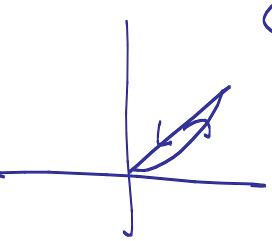
Green: $\iint_{\text{square}} (2-1)\, dA =$ area(square) $= \underline{4}$

e) $\oint_C xy^2dx + x^2ydy$, where C is given by $r(t) =< 4\cos(t), 2\sin(t) >$, t between 0 and 2 Pi.

Green, conservative vector field: $= \underline{0}$

f) $\int_C xydx + x^2dy$ where C is the boundary of the region between the graphs of $y = x^2$ and $y = x$.

Green: $= \iint_R 2x - x\, dA = \int_0^1 \int_{x^2}^{x} x\, dy\, dx =$ computer



10. Prove that if $F(x,y,z) =< M(x,y,z), N(x,y,z), P(x,y,z) >$ is any vector field where M, N, P are twice continuously differentiable then $div(curl(F)) = 0$

just try it, it will work out, i.e. $curl(F) = (N_y - P_z, \ldots)$

$div(curl(F)) = (\ \ )+(\ \ )+(\ \ ) = 0$

~~Use Green's Theorem to prove that integrals of a conservative vector fields over closed curves are zero (assuming that the closed curve encloses a simply connected region and all conditions of Green's theorem are satisfied).~~

Suppose a region R is enclosed by a simple, closed curve C. Use Green's theorem to show that $area(R) = \int_C \frac{x}{2}dy - \frac{y}{2}dx$. Use this result to find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$. *Hint: the boundary of that ellipse is given by* $r(t) =< a\cos(t), b\sin(t) >$, *where* $0 \leq t \leq 2\pi$

$$\oint_C \frac{x}{2}dy - \frac{y}{2}dx = \oint_C N dy + M dx = \iint_R N_x - M_y \, dA$$

$N = \frac{x}{2}$, $\Rightarrow N_x = \frac{1}{2}$

$M = -\frac{y}{2}$ $\Rightarrow M_y = -\frac{1}{2}$

Note: N goes with the dy, M goes with the dx

$$= \iint_R \frac{1}{2} + \frac{1}{2} \, dA = \iint_R dA = area(R)$$

If $r(t) = \langle a\cos(t), b\sin(t) \rangle$, $area(R) = \int_0^{2\pi} \frac{x}{2}dy - \frac{y}{2}dx =$

$$= \int_0^{2\pi} \frac{a}{2}\cos(t)\, b\cos(t)\,dt + \frac{b}{2}\sin(t)\, a\sin(t)\,dt$$

$$= \frac{ab}{2}\int_0^{2\pi} 1\,dt = ab\cdot\pi$$