Math 2511 – Calc III Practice Exam 1

This is a practice exam. The actual exam consists of questions of the type found in this practice exam, but will be shorter. If you have questions do not hesitate to send me email.

1. **Definitions**: Please state in your own words the meaning of the following terms:
   a) Vector
   b) Dot product, cross product
   c) Angle between two vectors
   d) Unit vector
   e) Line, plane
   f) Projection of vector a onto vector b
   g) Distance formulas between planes and points, lines and points, planes and lines, planes and planes, lines and lines
   h) Tangent vector to a curve
   i) Unit tangent vector to a curve
   j) Unit normal vector to a curve
   k) Binormal vector
   l) Curvature
   m) Length of a curve
   n) Integral of a curve
   o) Slinky, spirals

2. **True/False** questions:
   a) $u \cdot u = \|u\|^2$ T

   b) $< 1,3,2 >$ and $< -4,-2,5 >$ are perpendicular T $\langle 1,3,2\rangle \cdot \langle -4,-2,5\rangle = -4-6+10=0$

   c) $< 1,3,-2 >$ and $< 2,6,4 >$ are parallel F

   d) $v \cdot w = -w \cdot v$ F (true for $v \times w = -w \times v$

   e) $\frac{d}{dt}\|r(t)\| = \left\|\frac{d}{dt} r(t)\right\|$ F

   f) $\frac{d}{dt} p(t) \times r(t) = p'(t) \times r'(t)$ T

   g) $r(t) = \left\langle \sqrt{t}+2, 3-\sqrt[3]{t}, \sqrt[4]{t} \right\rangle$ is the equation of a line F

   h) If $\|r(t)\| \equiv 1$ then $r(t) \times r'(t) = 0$ F (but $r \cdot r' = 0$)

   i) The planes $x+3y+2z=5$ and $4x+2y-5z=0$ are perpendicular T $\langle 1,3,2\rangle \cdot \langle 4,2,-5\rangle = 4+6-10=0$

   j) The distance between $x-y+z=2$ and $x+y+z=1$ is zero
   Not parallel, so T

3. **Vectors**: Suppose $u = <7,-2,3>$, $v = <-1,4,5>$, and $w = <-2,1,-3>$
   a) Are u and v orthogonal, parallel, or neither?

$u \cdot v = \langle 7,-2,3\rangle \cdot \langle -1,4,5\rangle = -7-8+15=0$ so $\underline{u, v \text{ are orthog.}}$

   b) Find graphically and algebraically $2u+3v$ and $u-v$

$2u+3v = \langle 14,-4,6\rangle + \langle -3,12,15\rangle = \underline{\langle 11,8,21\rangle}$

$u-v = \underline{\langle 8,-6,-2\rangle}$

   c) Find the angle between v and w

$\cos(\theta) = \dfrac{v\cdot w}{\|v\|\,\|w\|} = \underline{-9/\sqrt{40}\cdot\sqrt{14}}$

$v\cdot w = \langle -1,4,5\rangle\cdot\langle -2,1,-3\rangle = 2+4-15=-9$, $\|v\| = \sqrt{40}$, $\|w\| = \sqrt{14}$

   d) Find $u \cdot v$ (dot product), $u \times v$ (cross product), $u \cdot (v \times w)$, and $\|u\|$

$\underline{u\cdot v = 0}$, $u\times v = \underline{\langle -22,-38,26\rangle}$, $u\cdot(v\times w) = \underline{-72}$

$\|u\| = \|\langle 7,-2,3\rangle\| = \sqrt{49+4+9} = \underline{\sqrt{62}}$

   e) Find the projection of w onto u and the projection of u onto w

$proj_{\vec{u}}(\vec{w}) = \dfrac{\langle 7,-2,3\rangle\cdot\langle -2,1,-3\rangle}{\|u\|^2}\cdot\langle 7,-2,3\rangle = \dfrac{-25}{62}\langle 7,-2,3\rangle$

$proj_{\vec{w}}(\vec{u}) = \dfrac{\langle 7,-2,3\rangle\cdot\langle -2,1,-3\rangle}{\|w\|^2}\langle -2,1,-3\rangle = \dfrac{25}{14}\langle -2,1,-3\rangle$

4. **Lines and Planes**
   a) Find the equation of the plane spanned by $<1,3,-2>$ and $<2,1,2>$ through the point $P(1,2,3)$

$n = \langle 1,3,-2\rangle \times \langle 2,1,2\rangle = \langle 8,-6,-5\rangle$

$\Rightarrow 8x-6y-5z+D=0$

$16-6-10+D=0 \Rightarrow \underline{D=0}$

$\underline{8x-6y-5z=0}$

b) Find the equation of the plane through $P(1,2,3)$, $Q(1,-1,1)$, and $R(3,2,1)$

$\vec{PQ} = \langle 0, -3, -2 \rangle$ $\quad n = \langle 0,-3,-2 \rangle \times \langle 2,0,-2 \rangle = \langle 6,-4,6 \rangle$

$PR = \langle 2, 0, -2 \rangle$ $\quad = 2\langle 3,-2,3 \rangle$

$3x - 2y + 3z + D = 0, \quad D = -8$

c) Find the equation of the plane parallel to $x - y + z = 2$ through $P(0,2,0)$

$n = \langle 1,-1,1 \rangle$

$\Rightarrow x - y + z + d = 0 \Rightarrow -2 + d = 0 \Rightarrow x - y + z + 2 = 0$

d) Find the equation of the line through $P(1,2,3)$ and $Q(1,-1,1)$

$\vec{v} = PQ = \langle 0,-3,-2 \rangle$

$l(t) = (1,2,3) + t\langle 0,-3,-2 \rangle = \langle 1, 2-3t, 3-2t \rangle$

e) Do the plane $x - y + z = 2$ and the line $l(t) = \langle 1+t, 2t, 1-5t \rangle$ intersect? If so, where?

$(1+t) - (2t) + (1-5t) = 2 \Rightarrow t = 0 \Rightarrow l(0) = \langle 1,0,1 \rangle$ point of intersection

$-6t = 0$

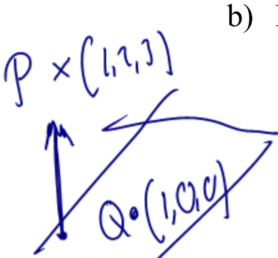
5. **Distances**

a) Find the distance between the line $x - y = 2$ and $P(1,2)$

$x - y - 2 = 0$

$d = \dfrac{|1-2-2|}{\sqrt{2}} = 3/\sqrt{2}$

b) Find the distance between the plane *x + y + z = 1* and the point *P(1, 2, 3)*



$PQ = \langle 0,-2,-3 \rangle$ $\quad$ or $\quad d = \dfrac{|1+2+3-1|}{\sqrt{3}} = 5/\sqrt{3}$

$d = \|\text{proj}_{\vec{n}} PQ\| = \dfrac{|\langle 1,1,1 \rangle \cdot \langle 0,-2,-3 \rangle|}{\sqrt{3}} = 5/\sqrt{3}$

c) Find the distance between the planes $x - y + z = 2$ and $2x - 2y + 2z = 5$

parallel, so dist $\neq 0$. $P(2,0,0)$ is on plane 1 $\quad d = \dfrac{|4-5|}{\sqrt{12}} = 1/\sqrt{12}$

6. **Vector valued functions**:

a) Find $r'(t)$ if $r(t) = < 6t, -7t^2, t^3 >$

$$r'(t) = \langle 6, -14t, 3t^2 \rangle$$

b) Find $r'(t)$ if $r(t) = < a\cos^3(t), a\sin^3(t), t\sin(t) >$

$$r'(t) = \langle 3a\cos^2(t)\cdot(-\sin(t)),\ 3a\sin^2(t)\cdot\cos(t),\ \sin(t) + t\cos(t) \rangle$$

c) If $r(t) = < 4t, t^2, t^3 >$, find $r'(t)$, $r''(t)$, $\frac{d}{dt}\|r(t)\|$

$$r'(t) = \langle 4, 2t, 3t^2 \rangle$$

$$r''(t) = \langle 0, 2, 6t \rangle$$

$$\|r(t)\| = \sqrt{16t^2 + t^4 + t^6} \Rightarrow \frac{d}{dt}\|r(t)\| = \frac{32t + 4t^3 + 6t^5}{2\sqrt{16t^2 + t^4 + t^6}}$$

d) If $r(t) = < e^t, 3t^3, \frac{3}{6t} >$ some curve, find $\int_1^2 r(t)dt$

$$\int_1^2 \langle e^t, 3t^3, \frac{3}{6t} \rangle dt = \langle \int_1^2 e^t dt, \int_1^2 3t^3 dt, \int_1^2 \frac{1}{2t} dt \rangle =$$

$$= \langle e^2 - e, \frac{3}{4}(2^4 - 1), \frac{1}{2}\ln(2) \rangle$$

e) If $r(t) = < t, \frac{1}{t} >$, find *T(t), N(t), and B(t)*

$$r'(t) = \langle 1, -1/t^2 \rangle$$

$$\|r'(t)\| = \sqrt{1 + 1/t^4} = \frac{\sqrt{t^4+1}}{t^2}$$

$$\Rightarrow T = \frac{t^2}{\sqrt{t^4+1}}\langle 1, -1/t^2 \rangle = \langle \frac{t^2}{\sqrt{t^4+1}}, -\frac{1}{\sqrt{t^4+1}} \rangle = \frac{1}{\sqrt{t^4+1}}\langle t^2, -1 \rangle$$

continued on last page

f) Repeat (e) for $r(t) = < e^t \cos(t), e^t \sin(t) >$ for $t = \frac{\pi}{2}$

$r'(t) = \langle e^t\cos(t) - e^t\sin(t), e^t\sin(t) + e^t\cos(t) \rangle = e^t\langle \cos(t) - \sin(t), \sin(t) + \cos(t) \rangle$

$\|r'(t)\| = e^t\sqrt{(\cos(t)-\sin(t))^2 + (\sin(t)+\cos(t))^2} = \sqrt{2}\,e^t$

$\Rightarrow T = \frac{1}{\sqrt{2}}\langle \cos(t)-\sin(t), \sin(t)+\cos(t) \rangle \Rightarrow T'(t) = \frac{1}{\sqrt{2}}\langle -\sin(t)-\cos(t), \cos(t)-\sin(t) \rangle$

$\Rightarrow \underline{T(\pi/2) = 1/\sqrt{2}\langle -1, 1 \rangle}$ $\qquad \Rightarrow \|T'\| = 1$

$\Rightarrow N = \frac{1}{\sqrt{2}}\langle -(\sin(t)+\cos(t)), \cos(t)-\sin(t) \rangle \Rightarrow \underline{N(\pi/2) = \frac{1}{\sqrt{2}}\langle -1, -1 \rangle}$

g) If $r(t) = < 3-3t, 4t >$, find the arc length of the curve between 0 and 1

$\underline{L} = \int_0^1 \|r'(t)\|\,dt = \int_0^1 \|\langle -3, 4 \rangle\|\,dt = \underline{5}$

h) If $r(t) = < 4t, 3\cos(t), 3\sin(t) >$, find the arc length of the curve between 0 and $\frac{\pi}{2}$

$r'(t) = \langle 4, -3\sin(t), 3\cos(t) \rangle$

$\Rightarrow \underline{L} = \int_0^{\pi/2} \|r'(t)\|\,dt = \int_0^{\pi/2} \sqrt{16 + 9(\sin^2(t) + \cos^2(t))}\,dt = \underline{5 \cdot \frac{\pi}{2}}$

i) Find the curvature of $r(t) = < t, 3t^2, \frac{t^2}{2} >$

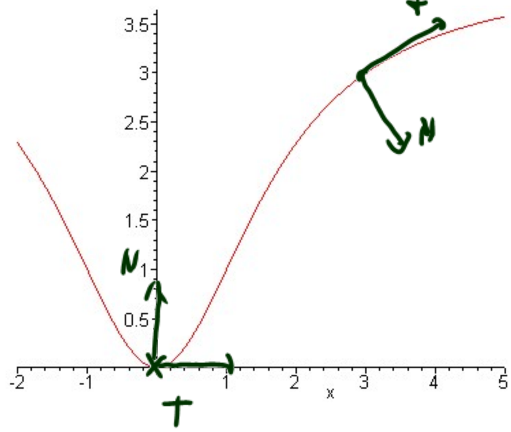
$r'(t) = \langle 1, 6t, t \rangle$

$r''(t) = \langle 0, 6, 1 \rangle$

$r' \times r'' = \begin{vmatrix} i & j & k \\ 1 & 6t & t \\ 0 & 6 & 1 \end{vmatrix} = \langle 0, -1, 6-6t \rangle$

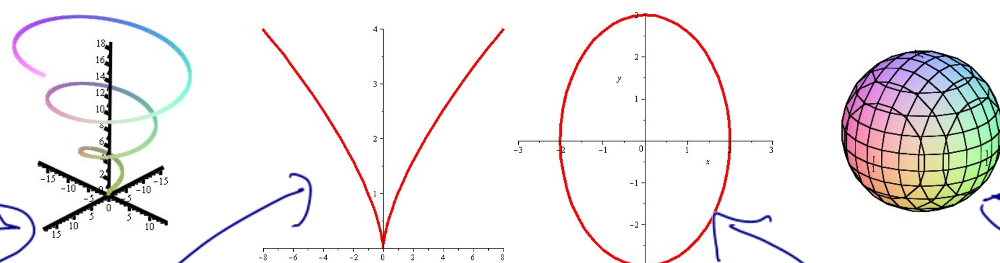
$\Rightarrow K = \frac{\|r' \times r''\|}{\|r'\|^3} = \frac{\sqrt{(6-6t)^2 + 1}}{(\sqrt{1+37t^2})^3}$

8. **Picture:** Sketch ~~the circle that fits the graph below the best~~ T, N at the points x = 0 and x = 3. At which of the two points is the curvature smaller?



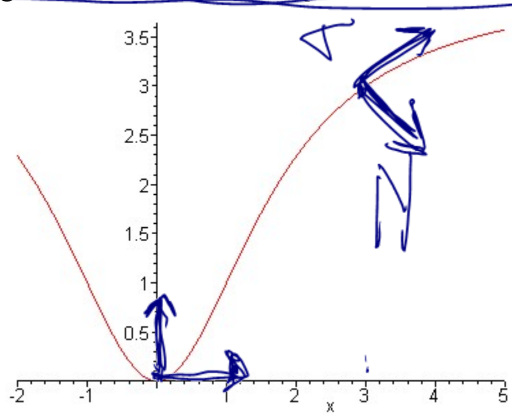
κ is smaller at $x = 3$

9. **Picture**: Match the following functions to their corresponding plots.



$r(t) =< t^3, t^2 >$ $\quad x^2 + y^2 + z^2 = 1$ $\quad r(t) =< t\cos(t), t\sin(t), t >$ $\quad r(t) =< 2\sin(t), 3\cos(t) >$

10. **Picture**: The graph below shows a vector-valued function. Sketch the unit tangent, unit normal, ~~acceleration, tangential and normal components of the acceleration~~ for $t = 3$. and $t = 0$



11. **Prove** the following facts:

a) Show that $u \times v = -(v \times u)$

$$u \times v = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \langle \cdots \rangle \qquad v \times u = \langle \cdots \rangle$$

then compare

b) Show that $u \cdot (v \times u) = 0$

$$\langle u_1, u_2, u_3 \rangle \cdot \left(\langle v_1, v_2, v_3 \rangle \times \langle u_1, u_2, u_3 \rangle \right) = \ldots$$

c) Show that if $y = f(x)$ is a function that is twice continuously differentiable, then the curvature of f at a point x is $K = \dfrac{|f''(x)|}{\left(1+[f'(x)]^2\right)^{3/2}}$

$y = f(x) \Rightarrow r(t) = \langle t, f(t) \rangle = \langle t, f(t), 0 \rangle$

$r' = \langle 1, f', 0 \rangle$

$r'' = \langle 0, f'', 0 \rangle$

$r' \times r'' = \langle 0, 0, f'' \rangle$

$\Rightarrow \chi = \dfrac{\|r' \times r''\|}{\|r'\|^3} = \dfrac{|f''(t)|}{\sqrt{1+[f'(t)]^2}^3}$

d) Prove that the curvature of a line in space is zero.

straight. forward!

$$T(t) = \frac{1}{\sqrt{t^4+1}} \langle t^2, -1 \rangle$$

$$\Rightarrow T'(t) = -\frac{1}{\cancel{2}} \frac{\cancel{2}4t^3}{(t^4+1)^{3/2}} \langle t^2, -1 \rangle + \frac{1}{(t^4+1)^{1/2}} \langle 2t, 0 \rangle =$$

$$\left\langle \frac{-2t^5}{(t^4+1)^{3/2}} + \frac{2t}{(t^4+1)^{1/2}}, \frac{2t^3}{(t^4+1)^{3/2}} \right\rangle = \frac{1}{(t^4+1)^{3/2}} \langle -2t^5 + 2t(t^4+1), 2t^3 \rangle$$

$$= \frac{2t}{(t^4+1)^{3/2}} \langle 1, t^2 \rangle$$

$$\Rightarrow \|T'\| = \frac{2t}{(t^4+1)^{3/2}} \cdot (1+t^4)^{1/2} = \frac{2t}{t^4+1}$$

$$\Rightarrow N = \frac{T'}{\|T'\|} = \frac{2t}{(t^4+1)^{3/2}} \langle 1, t^2 \rangle \cdot \frac{(t^4+1)}{2t} = \frac{1}{\sqrt{t^4+1}} \langle 1, t^2 \rangle$$

Note that $\|T\| = 1$, $\|N\| = 1$, and $T \cdot N = 0$ so this answer might even be right!