Summary 21: Polar Coordinates

Changing Coordinates to Polar Coordinates: To change from rectangular coordinates to polar coordinates:

1. Substitute $\boldsymbol{x = r\cos(\theta)}$ and $\boldsymbol{y = r\sin(\theta)}$
2. Change $dxdy$ or $dydx$ to $\boldsymbol{r\,dr\,d\theta}$ (note the extra r)
3. Find the bounds for the radius $\boldsymbol{r}$ and angle $\boldsymbol{\theta}$ so that the domain D is covered exactly once
4. Use the fact that $x^2 + y^2 = r^2\cos^2(\theta) + r^2\sin^2(x) = r^2$ to simplify the integrand if possible
5. Integrate as usual, treat r and θ as just two ordinary variables

Try this technique in problems that involve "round" domain such as circles or disks, and/or complicated integrands involving $x^2 + y^2$ terms

Compute $\iint_D 2x^2 + 2y^2\,dA$, where D is the set in the first quadrant outside the circle $x^2 + y^2 = 1$ but inside $x^2 + y^2 = 9$

We clearly have "round" domains here as well as an integrand involving $x^2 + y^2$ so polar coordinates seem advisable. Looking at the picture of the domain, we see that the angle goes between 0 and $\frac{\pi}{2}$ radians (to stay in the first quadrant) and the radius is between 1 and 3. Thus:

$$\iint_D 2x^2 + 2y^2 dA =$$

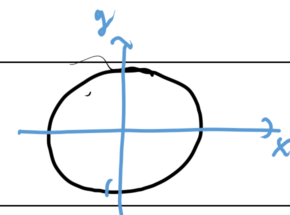
$$= \int_0^{\frac{\pi}{2}}\int_1^3 2r^2(\cos^2(\theta) + \sin^2(\theta))\boldsymbol{r}drd\theta =$$

$$= \int_0^{\frac{\pi}{2}}\int_1^3 2r^3\,drd\theta = \frac{\pi}{2}\left(\frac{2}{4}3^4 - \frac{2}{4}\right) = \frac{80\pi}{4} =$$

$$= 20\,\pi$$

Area of a Circle with radius R:

$$area(Circle) = \iint_D 1\,dA = \int_0^{2\pi}\int_0^R \boldsymbol{r}drd\theta = \frac{2\pi R^2}{2} = \pi R^2$$



Volume of a Sphere with radius R:

$$vol(Sphere) = 2\iint_{circle,radius\ R}\sqrt{R^2 - x^2 - y^2}\,dA =$$

$$= 2\int_0^{2\pi}\int_0^R \sqrt{R^2 - r^2}\,\boldsymbol{r}drd\theta =$$

$$= 2\cdot 2\pi\cdot\frac{2}{3}\cdot\left(-\frac{1}{2}\right)(R^2 - r^2)^{\frac{3}{2}}\Big|_{r=0}^{r=R} =$$

$$= -\frac{4}{3}\pi(0 - (R^2)^{\frac{3}{2}}) =$$

$$= \frac{4}{3}\pi R^3$$