Summary 19: Integration

Definition:

$$\iint_D f(x,y)dA = \lim_{n,m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i, y_j)\Delta x_i \Delta y_j.$$

Geometric Meaning:

$\iint_D f(x,y)dA$ gives the (net) volume under the surface $z = f(x,y)$ over the set D in the plane

How-to Theorem (Fubini's Theorem):

Suppose $f(x,y)$ is continuous on the rectangle $[a,b] \times [c,d]$. Then

$$\iint_D f(x,y)dA = \int_a^b \int_c^d f(x,y)\, dy\, dx = $$
$$= \int_c^d \int_a^b f(x,y)\, dx\, dy$$

Note: we typically setup the integral by hand, but we evaluate it using Mathematica. For the example on the right we would type:

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Integrate[Integrate[2 x y + 3 x^2 - y^2, {y, 0, 3}], {x, 0, 2}]
24

Integrate[Integrate[2 x y + 3 x^2 - y^2, {x, 0, 2}], {y, 0, 3}]
24
```

Find $\iint_D 2xy + 3x^2 - y^2\, dA$ *where* $D = [0,2] \times [0,3]$

Using Fubini's theorem, we have

$\iint_D 2xy + 3x^2 - y^2\, dA$ =

= $\int_0^2 \int_0^3 2xy + 3x^2 - y^2\, dy\, dx$ =

= $\int_0^2 \left(xy^2 + 3x^2y - \frac{1}{3}y^3 \Big|_{y=0}^{y=3} \right) dx$ =

= $\int_0^2 (9x + 9x^2 - 9)\, dx$ =

= $\frac{9}{2}x^2 + 3x^3 - 9x \Big|_0^2$ =

= $18 + 24 - 18 = 24$

And also:

$\iint_D 2xy + 3x^2 - y^2\, dA$ =

= $\int_0^3 \int_0^2 2xy + 3x^2 - y^2\, dx\, dy$ =

= $\int_0^3 (x^2y + x^3 - y^2x|_{x=0}^{x=2})\, dy$ =

= $\int_0^2 (4y + 8 - 2y^2)\, dy$ =

= $2y^2 + 8y - \frac{2}{3}y^3 \Big|_0^3$ =

= $18 + 24 - 18 = 24$

Integrating over more general domains:

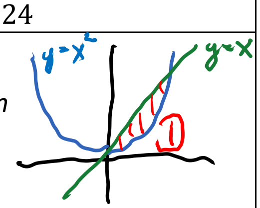
If $D = \{ (x,y): a \le x \le b, g_1(x) \le y \le g_2(x)\}$ then

$$\iint_D f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$

If $E = \{ (x,y): c \le y \le d, h_1(y) \le x \le h_2(y)\}$ then

$$\iint_E f(x,y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy$$

Find $\iint_D 2xy\, dA$ *where* D *is the region bounded by* $y = x^2$ *and* $y = x$



$\iint_D 2xy\, dA = \int_0^1 \int_{x^2}^{x} 2xy\, dy\, dx$ =

= $\int_0^1 \left(xy^2 |_{y=x^2}^{y=x} \right) dx$ =

= $\int_0^1 x\, x^2 - x\,(x^2)^2\, dx$ =

= $\int_0^1 x^3 - x^5\, dx = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$