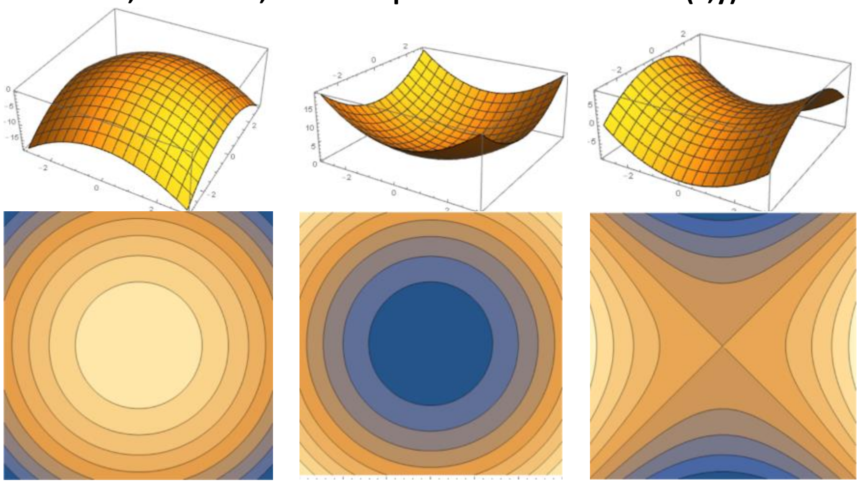
Summary 17: Local Extrema (Finding Min, Max, Saddle)

Maximum, minimum, or saddle point for a function z = f(x,y)



Max Min Saddle

How to find local extrema

1. Find the gradient ∇f
2. Solve system of equations $\nabla f = < 0,0 >$ to find critical points
3. Find Hessian matrix $H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$ at critical points
4. Find $D = \det(H) = f_{xx} f_{yy} - (f_{xy})^2$ at critical points

Then a critical point (x_0, y_0) is:

a) Min if $D > 0$ and $f_{xx} > 0$
b) Max if $D > 0$ and $f_{xx} < 0$
c) Saddle if $D < 0$
d) No info otherwise

Classify local extrema for

$$f(x,y) = x^4 + y^4 - 4xy + 1$$

$f_x = 4x^3 - 4y = 0$ so that $y = x^3$
$f_y = 4y^3 - 4x = 0$ so that $x^9 - x = 0$
Critical points: $(0,0), (-1,-1), (1,1)$
$H = \begin{pmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{pmatrix}$ and
$D = 144x^2y^2 - 16$ so that the point

- $(0,0)$: D is negative, so saddle
- $(1,1)$: $D > 0$, $f_{xx} > 0$, so min
- $(-1,-1)$: $D > 0, f_{xx} > 0$, is min