Summary 15: Partial Derivatives

Partials: For $f(x,y)$ define partial derivatives: $$\frac{\partial f}{\partial x} = f_x = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$ $$\frac{\partial f}{\partial y} = f_y = \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$$	*Use definition of partial derivative to find f_x for $f(x,y) = x^2y - 3xy$* $f_x = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h} =$ $= \lim_{h\to 0}\frac{((x+h)^2y-3(x+h)y)-(x^2y-3xy)}{h} =$ $= \lim_{h\to 0}\frac{x^2y+2xhy+h^2y-3xy-3hy-x^2y+3xy}{h} =$ $= \lim_{h\to 0}\frac{2xyh+h^2y-3hy}{h} =$ $= 2xy - 3y$
How to find Partials: To really find the partial derivative with respect to one variable, you consider all other variables as constant and take derivatives as usual.	*Find all partials for $f(x,y,z) = xysin(xz)$:* $f_x(x,y,z) = ysin(xz) + xycos(xz)z$ $f_y(x,y,z) = xsin(xz)$ $f_z(x,y,z) = xycos(xz)x$
Higher order Partials: A partial derivative is again a function of several variables, hence it again has partials. A function of two variables has 2 partials f_x, f_y, 4 2nd order partials $f_{xx}, f_{xy}, f_{yx}, f_{yy}$, 8 3rd order partials, etc. For "most" functions the order of taking partials does not matter, i.e. $f_{xy} = f_{yx}$ (this is called Clairaut's Theorem) We sometimes write $f_{xx} = \frac{\partial^2}{\partial x^2}$, $f_{xy} = \frac{\partial^2}{\partial x \partial y}$, etc.	*Find all 2nd order partials for $f(x,y) = \frac{y}{x}$:* $f_x = -\frac{y}{x^2}$ and $f_y = \frac{1}{x}$. Therefore $f_{xx} = \frac{\partial}{\partial x}\left(-\frac{y}{x^2}\right) = \frac{2y}{x^3}$ $f_{xy} = \frac{\partial}{\partial y}\left(-\frac{y}{x^2}\right) = -\frac{1}{x^2}$ $f_{yx} = \frac{\partial}{\partial x}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$ $f_{yy} = \frac{\partial}{\partial y}\left(\frac{1}{x}\right) = 0$
Partials graphically: f_x and f_y give the slope of f at (x,y) in the x and y direction, respectively. They can be used to define the **tangent plane** $$z = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + f(x_0,y_0)$$ Mathematica's `Plot3D` function can be used to visualize the tangent plane to a function f at a point (x_0,y_0)	*Tangent plane to $f(x,y) = x^2 + 2y^2$ at $(3,1)$:* $f_x = 2x$ so $f_x(3,1) = 6$ $f_y = 4y$ so $f_y(3,1) = 4$ Since $f(3,1) = 11$, tangent plane is: $$z = 6(x-3) + 4(y-1) + 11$$ `Plot3D[{x^2+2y^2, 6(x-3)+4(y-1)+11},` `{x,-4,4},{y,-4,4}]`